# Specific complex geometry of certain complex surfaces and three-folds

by

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# Contents

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# Declaration

The work in this thesis is original as far as I am aware, except where explicitly stated to the contrary. A part of the content of Chapter 1 was published in Serdica - Bulgaricae Mathematicae Publicationes, Vol. **14**, (1988), pp. 283 - 290.

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## Abstract

One of the most important consequences of Yau's proof of the Calabi's conjecture is the existence of a non-trivial Ricci-flat metric on K3 surfaces. For its explicit construction would be of great interest. Since it is not available yet the qualitative description of this metric would also have certain significance. In Chapter 1 we propose an approximation of the K3 Kähler-Einstein-Calabi-Yau metric for Kummer surfaces. It is obtained by gluing 16 pieces of the Eguchi-Hanson metric and 16 pieces of the Euclidean metric. Two estimates on its curvature are proved. Then we discuss the possibility of application of C.Taubes's iteration scheme for solving anti-self-duality equations. The reason is that the curvature of the metric in question is concentrated in small thin regions and it is almost anti-self-dual. It can be also used later to deduce stability of Kummer surfaces' tangent bundle.

In Chapter 2 we consider a special case of compact 3-folds $M$ which are diffeomorphic to the connected sum of $n$ copies of $S^3 \times S^3$. If $n \geq 103$, there is a complex structure of $c_1 = 0$ on $M$, which is a non-Kähler manifold. We prove that there are no non-trivial line bundles on $M$ and hence we deduce that its tangent bundle is stable with respect to any Gauduchon metric. By a theorem of Li and Yau we conclude that there is an Hermitian-Einstein metric on $M$. Our basic hypothesis is that the Hermitian-Einstein metric and the Gauduchon metric coincide. This is similar to the previous situation on K3. Then we consider the deformations of this metric, keeping the volume and the complex structure fixed. We seek the place of $M$ in the classification of almost Hermitian manifolds by Gray and Hervella and explore some sorts of conditions which can be imposed on $M$ and which can substitute the Kähler one. We also show that on Hermitian non-Kähler manifolds with $h^{2,0} = 0$ there are no non-zero anti-symmetric deformations of the complex structure.

# Chapter 1

# A construction of almost anti-self-dual metrics on Kummer surfaces

## 1.1 Introduction

The existence of Kähler-Einstein, or more generally Hermitian-Einstein, metrics plays an essential role in the study of compact complex manifolds. This is so for Yau's proof of the Calabi's conjecture. The latter states that if the first Chern class $c_1$ of a compact Kähler manifold M vanishes, then there exists a Kähler metric on it, which is Ricci-flat [7, 8]. Calabi has proved the uniqueness of such a metric and has suggested how to prove its existence [7]. In this direction one can ask the question if M admits Kähler-Einstein metrics with regard to the sign of $c_1(M)$. If $c_1 \leq 0$, Yau has given a positive answer to the above question [64] and this has led him to some new results in differential and algebraic geometry [63]. The case $c_1 > 0$ is not still completely investigated, but there is a serious recent progress due to Tian [52, 53, 54], Nadel [43], Tian and Yau [56], Aubin [1, 2]. Yau's theorem and further development of the problem for finding Kähler-Einstein metrics could be considered as a generalization of Riemann's Uniformization Theorem to higher dimensions.

One of the most important consequences of Yau's proof of the Calabi's conjecture is the existence of non-trivial Ricci-flat Kähler metric on K3 surfaces. By definition, a K3 surface is a 2-dimensional compact complex manifold whose first Betti number $b_1 = 0$ and whose first Chern class $c_1 = 0$. By a result of Siu [48] it follows that every K3 surface is Kähler . From this fact and Yau's proof one concludes that K3 surfaces admit Kähler non-trivial Ricci-flat metrics. They can be used in the investigation of the moduli space of K3 (see [57, 31]).

For the present K3 are the unique simply-connected compact manifolds on which such metrics exist. Another compact manifold which admits an Einstein vacuum metric is the torus $T$ in $C^n$. In this case the unique solution of Einstein's vacuum equations is the restriction to $T$ of the Euclidean flat metric, but it is not so interesting from a differential geometrical point of view. The explicit form of the K3 Kähler-Einstein-Calabi-Yau metric is not known yet. The problem of its constructive description was pointed out by Yau [65] and Kirby [30]. The construction of this metric in explicit form or in appropriate approximation is of great interest for both mathematicians [26] and physicists [45]. Actually, N.Hitchin has set in [26] the problem of finding of the K3 metric explicitly and proposed a method of attacking based on twistor theory. Later on, using also twistor ideas, Topiwala published a new proof of the Calabi's conjecture for Kummer surfaces [58, 59].

In this chapter we propose an approximation of the K3 metric in the particular case of a Kummer surface. It is constructed as follows.

Let $\Gamma \cong Z^4$ be a lattice in $C^2$, which is generated by four vectors, linearly independent over $R$. Consider the involution

$$\sigma : T \longrightarrow T$$

defined by

$$\sigma(x) = -x$$

which acts on the complex torus $T = C^2/\Gamma$. If we factorize the torus with respect to the relation of the equivalence

$$x \sim y$$

if and only if $\sigma(x) = y$, we shall obtain a singular surface

$$X = T/\sim = T/\sigma.$$

It is easy to see that $X$ has 16 singular points and near a singular point it can be embedded locally in $C^3$. In fact, near a singular point it can be identified locally with the cone $z^2 = xy$ in $C^3$.

Then we blow up the 16 singular points and let K be the resulting non-singular surface. K is said to be a Kummer surface. One verifies that $c_1(K) = 0$, and $b_1 = 0$ [49]. So K is certainly of type K3.

Under the sixteen $\sigma$ - processes every singular point is replaced by a copy of $CP^1$, the complex one-dimensional projective space. Inside a neighbourhood (ball) of every distinct projective line in K, which has radius $\lambda - \lambda^2$, where $\lambda$ is a sufficiently small number, we consider the metric of Eguchi- Hanson $g_{EH}$ ( see [9, 27, 15] ) and outside the neighbourhood of radius $\lambda$ - the Euclidean metric $g_E$. Let $(\alpha, \beta)$ be an appropriate partition of unity subordinate to the above balls. We define

$$h = \alpha g_{EH} + \beta g_E.$$

$h$ is a Hermitian metric, which is not Kähler one, but in large regions ( in those regions, where $h = g_{EH}$ or $h = g_E$ ) it is Kähler . Moreover, it is almost anti-self-dual because $g_{EH}$ and $g_E$ are anti-self-dual. The metric $h$ was deduced "heuristically" ( if we use the words of N.Hitchin,[26],p.115 ) by Page [46]. The purpose of the present chapter is to give a mathematically precise description of this metric, in particular to obtain estimates on its curvature (sections 1.2 - 1.4), and to discuss the possibility to be used for the proof of such geometrical properties as stability of the Kummer surface's tangent bundle. A part of the content covers the paper [5].

## 1.2 The curvature of $h$

We are going to introduce some notations and give the exact definition of the metric $h$ which is the main object in this chapter.

Let $A$ be the connection matrix of some connection on K. Then there is a well-defined operator

$$\nabla_A : \Gamma(\Lambda^p) \longrightarrow \Gamma(\Lambda^p \otimes \Lambda^1)$$

which is called covariant derivative. As usual $\Lambda^p$ denotes the space of exterior p-forms on K. Using the projection

$$\pi : \Gamma(\Lambda^p \otimes \Lambda^1) \longrightarrow \Gamma(\Lambda^{p+1})$$

one can define another operator

$$D_A = \pi \circ \nabla_A$$

which acts on the p-forms as follows

$$D_A \varphi = d\varphi + A \wedge \varphi + (-1)^{p+1} \varphi \wedge A.$$

Very often, instead of "connection $D_A$" ( or $\nabla_A$ ) we shall speak about the "connection A".

Now let $\varphi$ be a p-form on K ($0 \leq p \leq 4$). In some local coordinates $x^1, x^2, x^3, x^4$ it is expressed as

$$\varphi = \varphi_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Everywhere below we shall use Einstein's summation convention as above. Let $h$ be an Hermitian metric on K. $Re(h)$ is the corresponding Riemannian metric which we shall denote again by $h$ with no confusion. Both $h$ and the Euclidean metric determine Hodge operators $*_h$ and $*$. In local coordinates ( see [14] ) :

$$*\varphi = \frac{1}{p!} \epsilon^{i_1 \dots i_p}_{j_1 \dots j_{4-p}} \varphi_{i_1 \dots i_p} dx^{j_1} \wedge \dots \wedge dx^{j_{4-p}}$$

and

$$*_h \varphi = \frac{\sqrt{h}}{p!} h^{i_1 j_1} \dots h^{i_p j_p} \varphi_{i_1 \dots i_p} \epsilon_{j_1 \dots j_p j_{p+1} \dots j_4} dx^{j_{p+1}} \wedge \dots \wedge dx^{j_4}$$

where $h = \det(h_{ij})$ and $\epsilon$ is the fully antisymmetrical tensor. If $F$ is a matrix with elements the p-forms $\varphi^i_j$, then

$$|F|_h = \frac{1}{p!} h^{i_1 j_1} \dots h^{i_p j_p} \varphi^i_{j i_1 \dots i_p} \varphi^j_{i j_1 \dots j_p},$$

in particular

$$|F| = \frac{1}{p!} \varphi^i_{j i_1 \dots i_p} \varphi^j_{i i_1 \dots i_p}.$$

The $L_s$ norms of $F$ are defined by

$$\|F\|_{L_s} = \{\int_K |F|_h^{s/2}\sqrt{h}d^4x\}^{1/s}.$$

Let $\{p_j\}$,j=1,...,16, be the set of the singular points of the surface $X$:

$$p_j \in \{(0,0,0,0);(0,0,0,1/2);...;(1/2,1/2,1/2,1/2)\}.$$

Let $B_\lambda^j$ be the ball of radius $\lambda$ and centre at $p_j$. We choose

$$\lambda < \frac{1}{16}.$$

This condition provides

$$B_{4\lambda}^j \cap B_{4\lambda}^s = \emptyset$$

if $j \neq s$. Define the function $\alpha_j$ on $X$ by

$$\alpha_j(x) = \begin{cases} 1 & \text{if } x \in B_{\lambda-\lambda^2}^j \\ 0 & \text{if } x \notin B_\lambda^j \end{cases}$$

and $0 \leq \alpha_j \leq 1$. Set

$$\alpha(x) = \sum_{j=1}^{16} \alpha_j(x)$$

and $\beta(x) = 1 - \alpha(x)$, i.e. $\alpha + \beta = 1$. Then

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in B_{\lambda-\lambda^2}^j \text{ for some } j = 1,...,16; \\ 0 & \text{if } x \notin B_\lambda^s \text{ for every } s. \end{cases}$$

After making 16 $\sigma$-processes, we shall denote again by $B_\lambda^j$ the image of the ball $B_\lambda^j$ with no confusion. In the ball $B_{\lambda-\lambda^2} = B_{\lambda-\lambda^2}^j$ the metric of Eguchi-Hanson has the form

$$g_{EH} = (\frac{\sqrt{1+ct}}{(1+|z_1|^2)^2} + \frac{c|z_1|^2|z_2|^2}{\sqrt{1+ct}})dz_1 \otimes d\bar{z}_1 + \frac{c(1+|z_1|^2)\bar{z}_1 z_2}{2\sqrt{1+ct}}dz_1 \otimes d\bar{z}_2$$

$$+\frac{c(1+|z_1|^2)z_1\bar{z}_2}{2\sqrt{1+ct}}dz^2 \otimes d\bar{z}_1 + \frac{c(1+|z_1|^2)^2}{4\sqrt{1+ct}}dz_2 \otimes d\bar{z}_2,$$

where

$$t = (1+|z_1|^2)^2|z_2|^2, z_1, z_2 \in C,$$

$c > 0$ is an arbitrary constant ([9]). In order to prove a technical lemma (see Section 1.3) we choose $c = 4$. $g_{EH}$ is a Kähler metric on the bundle $L \longrightarrow CP^1$, where $L$ is biholomorphically equivalent to the cone

$$\{(x, y, z) \in C^3 : z^2 = xy\}.$$

See [9]. On the other hand our Kummer surface K defines the same bundle, that is $g_{EH}$ is the metric near every singular point $p$ we need. Then the metric $h$ is defined by

$$h = \alpha(x)g_{EH} + \beta(x)g_E,$$

where $g_E$ is the Euclidean metric. Let $(\varphi, U)$ be a local coordinate chart such that

$$p \in U, \;\; \varphi : U \longrightarrow R^4, \;\; \varphi(p) = 0.$$

Introduce real normal (for $h$) coordinates $x^1, x^2, x^3, x^4$. Thus

$$h^{ij}(p) = \delta^{ij}, \;\; dh^{ij}(p) = 0; \;\; \varphi(q) = x, q \in U, \;\; |h^{ij} - \delta^{ij}| \leq |\varphi|^2\rho(p) \leq |x|^2\rho(p).$$

for all $q \in U$. $|x|^2$ is the Euclidean norm of $x \in R^4$ and $\rho(p)$ is a constant which does not depend on $\lambda$. Hence

$$h^{ij} = \delta^{ij} + O(\lambda^2). \tag{1.1}$$

We shall work always in a neighbourhood of a blown up singular point $p = p_j$, i.e. it is sufficient to prove the estimates on the curvature of $h$ only in the ball $B_\lambda$ of radius $\lambda$, because outside this ball $h = g_E$ and here we have nothing to prove. We make the change of variables

$$z_1 = x^1 + \sqrt{-1}x^2, \quad z_2 = x^3 + \sqrt{-1}x^4.$$

The metric $h = \alpha g_{EH} + \beta g_E$ can be expressed in the form

$$h = (h_{ij}) = \begin{pmatrix} A & 0 & C & -D \\ 0 & A & D & C \\ C & D & B & 0 \\ -D & C & 0 & B \end{pmatrix}$$

where

$$A = \alpha\left(\frac{\sqrt{1+4t}}{(1+x^{1^2}+x^{2^2})^2} + \frac{4(x^{1^2}+x^{2^2})(x^{3^2}+x^{4^2})}{\sqrt{1+4t}}\right) + \beta,$$

$$B = \alpha\frac{(1+x^{1^2}+x^{2^2})^2}{\sqrt{1+4t}} + \beta.$$

$$C = 2\alpha\frac{(1+x^{1^2}+x^{2^2})(x^1x^3+x^2x^4)}{\sqrt{1+4t}},$$

$$D = 2\alpha\frac{(1+x^{1^2}+x^{2^2})(x^2x^3-x^4x^1)}{\sqrt{1+4t}}$$

and

$$t = (1+x^{1^2}+x^{2^2})^2(x^{3^2}+x^{4^2}).$$

If we look at the proof of **Lemma 1.1** carefully, we can get immediately (1.1) and (1.2), that is, our coordinates are actually normal.

Let $\nabla_{A_0}$ denote the Levi-Civita connection corresponding to $h$, which is determined by the Christoffel symbols

$$\Gamma^i_{jk} = h^{im}\left(\frac{\partial h_{km}}{\partial x^j} + \frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)/2,$$

where

$$h^{-1} = (h^{ij}) = \frac{1}{AB-C^2-D^2}\begin{pmatrix} B & 0 & -C & D \\ 0 & B & -D & -C \\ -C & -D & A & 0 \\ D & -C & 0 & A \end{pmatrix} \tag{1.2}$$

is the inverse matrix of $h$ and let $F_{A_0}$ denote the curvature of the connection $A_0$.

In section 1.4 we shall estimate the $L_p$-norms of $F_{A_0}$ and its self-dual part. For this purpose we need first to estimate $|F_{A_0}|_h$. This can be done by using the following lemma whose proof is in the next section.

**Lemma 1.1.** *There exists a constant $k_0 > 0$ independent of $\lambda$ such that in the ball $B_\lambda$ the estimate*

$$|R^i_{jkl}| \le k_0$$

*holds for all $i,j,k,l = 1,2,3,4$. $R^i_{jkl}$ are the components of the Riemann tensor of $h$.*

## 1.3 The main technical lemma

We are going to prove **Lemma 1.1.** It is well known that

$$R^i_{jkl} = \frac{1}{2}h^{is}\{\frac{\partial^2 h_{sk}}{\partial x_l \partial x_j} - \frac{\partial^2 h_{jk}}{\partial x_l \partial x_s} - \frac{\partial^2 h_{sl}}{\partial x_j \partial x_k} + \frac{\partial^2 h_{jl}}{\partial x_k \partial x_s}\}$$

$$+\frac{1}{4}h^{is}h^{tp}\{(\frac{\partial h_{tk}}{\partial x_s} + \frac{\partial h_{ts}}{\partial x_k} - \frac{\partial h_{sk}}{\partial x_t})(\frac{\partial h_{pj}}{\partial x_l} + \frac{\partial h_{pl}}{\partial x_j} - \frac{\partial h_{jl}}{\partial x_p})$$

$$-(\frac{\partial h_{tl}}{\partial x_s} + \frac{\partial h_{ts}}{\partial x_l} - \frac{\partial h_{ls}}{\partial x_t})(\frac{\partial h_{pj}}{\partial x_k} + \frac{\partial h_{pk}}{\partial x_j} - \frac{\partial h_{kj}}{\partial x_p})\}.$$

We see that it is sufficient to obtain upper estimates on the quantities

$$|h^{ij}|, \quad |\frac{\partial h_{ij}}{\partial x^k}|, \quad \text{and} \quad |\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l}|.$$

Everywhere below we assume that $x \in B_\lambda$, that is,

$$|x|^2 = \sum_{j=1}^{4} x_j^2 \leq \lambda^2 < \frac{1}{(16)^2} < 1.$$

1. First we shall estimate $|h^{ij}|$.

$$|D| \leq 2|\alpha|(1 + x_1^2 + x_2^2)(x_2 x_3 + x_4 x_1)/\sqrt{1+4t} \leq 2 \tag{1.3}$$

Similarly we obtain

$$|C| \leq 2, \quad |B| \leq 5, \quad |A| \leq 22. \tag{1.4}$$

We also have

$$AB - C^2 - D^2 = \alpha^2 + \beta^2$$

$$+\alpha\beta[\sqrt{1+4t}(1 + x_1^2 + x_2^2)^2 + (4(x_1^2 + x_2^2)(x_3^2 + x_4^2) + (1 + x_1^2 + x_2^2)^2)/\sqrt{1+4t}]. \tag{1.5}$$

Therefore

$$AB - C^2 - D^2 \geq \alpha^2 + \beta^2 = 2\alpha^2 - 2\alpha + 1 \geq 1/4. \tag{1.6}$$

From (1.2), (1.3), (1.4) and (1.6), it follows that

$$|h^{ij}| \leq C'$$

for some constant $C' > 0$.independent of $\lambda$

2. We estimate the second derivatives as follows.

$$\frac{\partial D}{\partial x_k} = 2\frac{\partial \alpha}{\partial x_k}(1 + x_1^2 + x_2^2)(x_2x_3 - x_4x_1)/\sqrt{1+4t}$$

$$+2\alpha[u\sqrt{1+4t} - 2(1 + x_1^2 + x_2^2)(x_2x_3 - x_4x_1)v/\sqrt{1+4t}]/(1+4t)$$

where

$$u = \frac{\partial}{\partial x_k}[(1 + x_1^2 + x_2^2)(x_2x_3 - x_4x_1)]$$

and

$$v = \frac{\partial}{\partial x_k}[(1 + x_1^2 + x_2^2)(x_3^2 + x_4^2)].$$

Then

$$|\frac{\partial D}{\partial x_k}| \leq 2\lambda^2|\frac{\partial \alpha}{\partial x_k}| + 2\alpha(|u|(1 + 4\lambda) + |v|).$$

It is easy to verify that $|u| \leq 1$ and $|v| \leq 1$. Hence

$$|\frac{\partial D}{\partial x_k}| \leq 2\lambda^2|\frac{\partial \alpha}{\partial x_k}| + 6.$$

In order to estimate the first and the second derivative of $\alpha$ we need the following

**Lemma 1.2.** ([29]) *The $C^\infty$ function $\alpha$ can be chosen such that there is a positive constant $L$, independent of $\lambda$, for which the estimates*

$$|\frac{\partial \alpha}{\partial x_k}| \leq L^2\lambda^{-2}$$

$$|\frac{\partial^2 \alpha}{\partial x_k \partial x_l}| \leq 4L^3\lambda^{-4}$$

*hold.*

Then

$$|\frac{\partial D}{\partial x_k}| \leq 2L^2 + 6.$$

Similarly we get that

$$|\frac{\partial C}{\partial x_k}| \leq 2L^2 + 6.$$

Further

$$\frac{\partial B}{\partial x_k} = \frac{\partial \alpha}{\partial x_k}[(1 + x_1^2 + x_2^2)^2 - \sqrt{1+4t}]/\sqrt{1+4t}$$

$$+\alpha[(\partial/\partial x_k)(1+x_1^2+x_2^2)^2.\sqrt{1+4t}-2v(1+x_1^2+x_2^2)^2/\sqrt{1+4t}]/(1+4t).$$

$$|\frac{\partial B}{\partial x_k}| \le |\frac{\partial \alpha}{\partial x_k}|.|(1+x_1^2+x_2^2)^2-\sqrt{1+4t}|+9.$$

We have

$$|(1+x_1^2+x_2^2)^2-\sqrt{1+4t}| \le 3\lambda^2+\sum_{k=1}^{\infty}(16\lambda^2)^k$$

since $t \le 4\lambda^2$. But we have chosen $\lambda < 1/16$ and therefore

$$|\frac{\partial B}{\partial x_k}| \le (3\lambda^2+32\lambda^2)|\frac{\partial \alpha}{\partial x_k}|+9.$$

From this and from **Lemma 1.2** we find

$$|\frac{\partial B}{\partial x_k}| \le 35L^2+9.$$

Analogously

$$|\frac{\partial A}{\partial x_k}| \le const.$$

In conclusion we obtain that

$$|\frac{\partial h_{ij}}{\partial x_k}| \le C''$$

for some $C'' > 0$ which is independent of $\lambda$.

3. After twice differentiating and applying **Lemma 1.2** we conclude that there is a constant $C''' > 0$ which is independent of $\lambda$ such that

$$|\frac{\partial^2 h_{ij}}{\partial x_k \partial x_l}| \le C'''/\lambda^{-2}.$$

The proof of this fact is lengthy but we omit it because there are no new points. Now we have estimated all terms of the formula in the beginning. Hence the desired estimate on the Riemannian tensor is a straightforward consequence from the estimates obtained in the points 1, 2 and 3.

## 1.4 A discussion of a possible application of Taubes's method

After the preparatory work in the previous two sections, we are now in position to prove our main result in this chapter.

**Proposition 1.1.** *There exists a constant $c > 0$ such that for all $p \geq 1$ and $\lambda < \frac{1}{16}$ the following estimates hold*

$$\|F_{A_0}\|_{L_p} \leq c\lambda^{\frac{4}{p}-2} \tag{1.7}$$

$$\|P_+F_{A_0}\|_{L_p} \leq c\lambda^{\frac{5}{p}-2}. \tag{1.8}$$

Here $P_+ = (1 + *_h)/2$ and $c$ is independent of $\lambda$.

**Proof.** The $L_p$-norm of $F_{A_0}$ is given by

$$\|F\|_{L_p} = \{\int_K |F|_h^{p/2}\sqrt{h}d^4x\}^{1/p}. \tag{1.9}$$

First we have

$$|F_{A_0}|_h \leq |h^{ke}||h^{lf}||R^i_{jkl}||R^j_{ief}|.$$

Then from **Lemma 1.1** and its proof we get that

$$|F_{A_0}|_h \leq k^2\lambda^{-4}, \tag{1.10}$$

where the constant $k$ is independent of $\lambda$. We also need to estimate $\sqrt{h}$. From (1.5) we obtain that

$$\sqrt{h} = (AB - C^2 - D^2) \leq 8 \tag{1.11}$$

in the ball $B_\lambda$. The exterior of the sixteen balls $B_\lambda$ does not give any contribution to the $L_p$-norm of the curvature since there $h = g_E$, that is, $h$ is flat. Then (1.9), (1.10) and (1.11) imply

$$\|F_{A_0}\|_{L_p} \leq const.\lambda^{-2}\{\int_{|x|<\lambda} d^4x\}^{1/p} \leq c\lambda^{\frac{4}{p}-2}.$$

In order to obtain the second estimate we note that in the 16 balls $B^j_{\lambda-\lambda^2}$ the metric $h$ coincides with the metric of Eguchi-Hanson. But the latter one is an anti-self-dual metric. Therefore $P_+F_{A_0} = 0$ in $B^j_{\lambda-\lambda^2}$ for any $j = 1, ..., 16$. It also vanishes outside the balls $B^j_\lambda$, since in this region $h$ coincides with the Euclidean flat metric as we have just mentioned. Hence the integration in the $L_p$-norm of $P_+F_{A_0}$ reduces to the overlapping area which consists of the sixteen rings $B^j_\lambda \setminus B^j_{\lambda-\lambda^2}$. Thus we get

$$\|P_+F_{A_0}\|_{L_p} \leq const.\lambda^{-2}\{\int_{\lambda-\lambda^2}^{\lambda} r^3dr\}^{1/p} \leq const.\lambda^{-2}(\lambda^4 - (\lambda-\lambda^2)^4)^{1/p} \leq c\lambda^{\frac{5}{p}-2}.$$

We note that (1.8) gives a little bit better estimate than (1.7). From (1.7) for $p = 2$ we see that the $L_2$-norm of the curvature of $A_0$ is bounded by a constant, which does not depend on $\lambda$. For $p = 2$ the estimate (1.8) gives

$$\|P_+F_{A_0}\|_{L_2} \leq c\sqrt{\lambda}.$$

Hence, we can make the $L_2$-norm of the self-dual part of the curvature sufficiently small choosing $\lambda$ to be sufficiently small. For this reason we would like to propose a

**Definition.** *A connection $A$, such that the $L_2$-norm of the self-dual part of the curvature of $A$ is bounded from above by a sufficiently small number, is called almost anti-self-dual connection.*

Therefore, according to this definition, the connection $A_0$, determined by $h$, is an almost anti-self-dual connection. Moreover, in large regions of the Kummer surface K it is in fact anti-self-dual.

Before concluding this section, we would like now to discuss the possibility of application of C.Taubes's iteration scheme [50, 51, 37] for solving anti-self-duality equations. The essential point of the method is the following.

One is looking for an anti-self-dual connection in the form

$$A = A_0 + a, \tag{1.12}$$

whose curvature satisfies the anti-self-duality equations

$$P_+F_A = 0. \tag{1.13}$$

The connection $A_0$ is fixed and $a$ is an unknown tensor. From (1.12) and (1.13) it follows that

$$\mathcal{D}_{A_0}\mathcal{D}^*_{A_0}u + \mathcal{D}^*_{A_0}u\sharp\mathcal{D}^*_{A_0}u = -P_+F_{A_0}, \tag{1.14}$$

where $a = \mathcal{D}^*_{A_0}u$ and notations are the same as in [50].

C.Taubes has solved the equations (1.14) by an iterative scheme [50]. The parameters of this procedure are in the terms of $\|F_{A_0}\|_{L_p}$, $\|P_+F_{A_0}\|_{L_p}$ and $\mu(A_0)$ - the first non-zero eigenvalue of the operator $\mathcal{D}_{A_0}\mathcal{D}^*_{A_0}$.

In order to have a convergent power series, which represents the solution of (1.14), the iteration parameters must satisfy some relations. This is so for an "appropriate" connection $A_0$, a priori constructed and depending on a sufficiently small parameter $\lambda$. We would also like to emphasize the important role of topology of the manifold M on which one solves the anti-self-duality equations (1.14). In the first version of Taubes's method [50] M has positive-definite intersection form, while in the improved variant by Donaldson ([13]), the so-called alternating method, the intersection matrix may have at most (up to orientation) two minus signs. The K3 is not of this type since $b_2^+ = 3$. Nevertheless, we have hoped that the Levi-Civita connection of the metric $h$, introduced in the section 1.2, could be used to produce the "appropriate" initial connection $A_0$ in the second version of Taubes's scheme [51], which covers also the case $b_2^+ = 3$. The reason for this is the fact that its curvature is concentrated in small thin regions of the Kummer surface. Moreover, it is almost anti-self-dual.

The **Proposition 1.1** provides two estimates on the curvature of $A_0$ and its self-dual part which are similar to those in [50]. However, one needs an estimate slightly better than (1.8) to apply Taubes's method, which for the present we have not achieved.

## 1.5 On the stability of the tangent bundles of Kummer surfaces

We shall prove stability of the tangent bundle of the Kummer surface K, supposing that there is an anti-self-dual connection $A = A_0 + a$, that is,

$$*_h F_A = -F_A. \tag{1.15}$$

Here $A_0$ is the Levi-Civita connection of $h$ and $A$ could be obtained by Taubes's method (see the previous section). Choose a basis $dx_1, dx_2, dx_3, dx_4$ of $T^*$ which is orthonormal at a given point with respect to the metric $h$. Denote

$$f_1^{\pm} = dx_1 \wedge dx_2 \pm dx_3 \wedge dx_4,$$

$$f_2^{\pm} = dx_1 \wedge dx_3 \pm dx_4 \wedge dx_2,$$

$$f_3^{\pm} = dx_1 \wedge dx_4 \pm dx_2 \wedge dx_3.$$

$(f_1^+, f_2^+, f_3^+)$ forms a basis of $\wedge_+^2$ - the space of self-dual 2 forms and $(f_1^-, f_2^-, f_3^-)$ is a basis of $\wedge_-^2$ -the space of anti-self-dual 2 forms ([14]). Since $F_A$ is anti-self-dual ((1.16)), we have

$$F_A = Mf_1^- + Nf_2^- + Pf_3^-$$

for some functions $M, N, P$. Introduce a complex basis $dz_1, dz_2$ of $T^{*1,0}(K) \cong T^*(K)$ (and therefore $d\bar{z}_1, d\bar{z}_2$ is a basis of $T^{*0,1}(K)$) by

$$dz_1 = dx_1 + idx_2,$$

$$dz_2 = dx_3 + idx_4.$$

Then we get that

$$f_1^- = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2),$$

$$f_2^- = \frac{1}{2}(dz_1 \wedge dz_2 - d\bar{z}_2 \wedge d\bar{z}_1)$$

$$f_3^- = \frac{i}{2}(dz_1 \wedge dz_2 + d\bar{z}_2 \wedge d\bar{z}_1)$$

and

$$*(dz_1 \wedge d\bar{z}_1) = dz_2 \wedge d\bar{z}_2,$$

$$*(dz_2 \wedge d\bar{z}_2) = dz_1 \wedge d\bar{z}_1,$$

$$*(dz_1 \wedge dz_2) = -dz_1 \wedge dz_2,$$

$$*(d\bar{z}_2 \wedge d\bar{z}_1) = -d\bar{z}_2 \wedge d\bar{z}_1.$$

Hence

$$F_A = \frac{iM}{2} dz_1 \wedge d\bar{z}_1 - \frac{iM}{2} dz_2 \wedge d\bar{z}_2 + \ldots$$

and

$$* F_A = \frac{iM}{2} dz_2 \wedge d\bar{z}_2 - \frac{iM}{2} dz_1 \wedge d\bar{z}_1 + \ldots \qquad (1.16)$$

Let $\omega$ be the fundamental form of $h$. In our basis it can be expressed as

$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

Define the operator $L$ by

$$L(\eta) = \omega \wedge \eta$$

and let $\Lambda$ be its $L_2$ adjoint. Then

$$\Lambda = L_* = w * L*$$

(see [62]). We want to compute $\Lambda F_A$. From (1.17) we obtain

$$L * F_A = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \wedge (\frac{iM}{2})(dz_2 \wedge d\bar{z}_2 - dz_1 \wedge d\bar{z}_1)$$

$$= -\frac{M}{4}(dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 - dz_2 \wedge d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_1) = 0.$$

Therefore $\Lambda F_A = 0$ and in this way we have proved the following

**Proposition 1.2.** *The cotangent bundle $T^*$ of the Kummer surface $K$ admits an Hermitian-Einstein connection.*

**Remark.** The definition of Hermitian-Einstein connection, degree of a sheaf, stability property, etc. can be found in Chapter 2.

Now the proof of stability of the cotangent (or tangent) bundle is straightforward. Indeed, by a conformal change

$$h_1 = fh,$$

where $f > 0$, we can get a new Hermitian metric $h_1$, whose fundamental form $\omega_1$ is $\partial\bar{\partial}$-closed:

$$\partial\bar{\partial}\omega_1 = 0.$$

See [21].This enables us to define the degree of the (holomorphic) cotangent bundle $T^*$ of K. Under conformal changes the Hodge $*$-operator is invariant and therefore the connection $A$ is anti-self-dual with respect to the metric $h_1$. Since the canonical class of our Kummer surface is trivial, the degree must vanish. On the other hand, according to **Proposition 2.2**, there is a Hermitian-Einstein connection on K and from $\deg(T^*) = 0$ one can obtain in the same way as in page 34 that its Einstein factor is 0. The latter is compatible with $\Lambda F_A = 0$. Now if we repeat the arguments of Lübke [41], we deduce that $T^*$ is semi-stable. But it is indecomposable since we are on K3. Therefore it is stable.

In conclusion we would like to note that if $Pic(K3) = 0$, that is, if there are no non-trivial line bundles on the considered K3 surface, then one can get the stability easily - exactly in the same way as in the section 2.4 of Chapter 2.

# Chapter 2

# On the complex structures of $\#_n S^3 \times S^3$

## 2.1 Introduction

Many aspects of the classification theory of complex three-dimensional manifolds have been clarified mainly due to the celebrated Mori programme. In this direction and especially for the search of a natural generalization of the K3 surfaces to higher dimensions the compact three-folds with trivial canonical bundle occupy an important position within the general scheme. Among the examples of such manifolds there is one class of particular interest. They are compact 3-folds $M$ with the following Hodge numbers:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

and whose canonical class $K_M = 0$([47, 18]). Miles Reid has called manifolds of this type "rakshasa" [47].

The first question which could be raised is whether such $M$ exist. Using a certain algebraic-geometrical procedure and C.T.C.Wall's classification of 6-manifolds [61] one can construct a complex structure $J$ of the above type on the connected sum of n copies of $S^3 \times S^3$. Since this is an important and quite intrigueing point, we shall give a brief description of the construction of $J$. The details and references can be found in [17, 18, 47].

One starts with a smooth quintic three-fold $N$ in $CP^4$ which contains infinitely many (-1,-1) smooth rational pairwise disjoint curves $C_i$, one of which is a line. Recall that a (-1,-1) curve $C$ in $N$ is a curve, isomorphic to $CP^1$, such that the normal bundle of $C$ in $N$ splits into $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The existence of such quintic threefolds $N$ is due to Clemens ([10]). In [18], p. 29, Friedman describes a modification of Clemens' construction which provides a simply connected $N$ such that $[C_i]$ span $H^2(N, \Omega^2)$ and there is a relation $\sum_i \lambda_i [C_i] = 0$ in $H^2(N, \Omega^2)$, where $\lambda_i \neq 0$ for every $i$. We shall omit this construction pointing out only that a K3 surface is involved in it.

Then we take $k \geq 2$ such curves $C_i$ of degrees $d_i$, one of which we choose to be a line. Since the $C_i$ are (-1,-1) disjoint curves, they can be contracted to $k$ ordinary double points $P_i$. In this way a three-fold $\bar{N}$ is obtained. Here by contraction we mean an isomorphism $N \setminus C_i \to \bar{N} \setminus P_i$ between complex analytic varieties. By [17] $\bar{N}$ has small deformations $M$ in which the singularities disappear and if $H^1(N, \mathcal{O}) = 0$, then all smoothings $M$ have trivial canonical bundle. From Lemma 8.1, [18], p. 25, $\pi_1(N) = \pi_1(M)$ and hence $M$ is simply connected. Moreover, since by the construction of $N$ the curves $C_i$ satisfy the above mentioned relation in $H^2(N, \Omega^2)$, the Corollary 8.8, p. 28 in Friedman's report [18] implies that $H_2(M, Z) = Z/dZ$, where $d$ is the greatest common divisor of the $d_i$. But $d = 1$ because one of the contracted curves is a line. Thus $H_2(M, Z) = 0$.

The Betti numbers of $M$ and $N$ are related by

$$b_2(M) = b_2(N) - s$$

and

$$b_3(M) = b_3(N) + 2k - 2s,$$

where $k - s$ is the rank of the kernel of $\bigoplus Z[C_i] \to H_2(N, Z)$ (see [18] or [47]). As we saw $b_2(M) = 0$. From the last formula and from the special construction of the "generic" quintic manifold $N$, the third Betti number of $M$ is in fact $b_3(M) = 2(k + 101)$. It can also be seen that $H_3(M, Z)$ is torsion-free.

Summarizing, this complicated algebraic-geometrical procedure provides a compact simply connected 6-manifold $M$ with $H_2(M, Z) = 0$, $H_3(M, Z)$ - torsion-free and which possesses a complex structure $J$ with trivial canonical class.

On the other hand, according to the classification of C.T.C.Wall [61] any compact oriented 6-manifold which is simply connected and whose second Stiefel-Whitney class $w_2 = 0$, is classified up to diffeomorphism by the third Betti number $b_3$, $H^2(Z)$, first Pontrjagin class $p_1$ and a trilinear map $H^2(Z) \times H^2(Z) \times H^2(Z) \to Z$ given by cup product. Restricting to the case $H^2(Z) = 0$, this implies that any simply connected manifold with $H^2(Z) = 0$ and $H_3(Z)$ - a torsion-free $Z$ module of rank $2n$ is diffeomorphic to a connected sum of $n$ copies of $S^3 \times S^3$ ([61]). Hence, since $w_2(M) = p_1(M) = 0$, $M$ is diffeomorphic to $\#_n S^3 \times S^3$ , where $n = 101 + k \geq 103$ and there is a complex structure $J$ on $M$, such that its first Chern class $c_1(J) = 0$. Moreover, there is also a special case of the Wall's result [61], which states that a compact simply connected 6-manifold with $H^*(Z)$ - torsion-free and $w_2 = 0$ has an almost complex structure if and only if $w_3 = 0$. In the latter case there is a unique up to homotopy almost complex structure with $c_1 = 0$. Therefore, as an almost complex structure, $J$ is unique up to homotopy.

R.Friedman [18] asks the question what is the minimal $n$ such that there exists a complex structure with trivial canonical bundle on the connected sum of $n$ copies of $S^3 \times S^3$. As we saw, it is at most 103. Probably another concrete example of a three-fold with trivial canonical class will reduce this number. Note that the Calabi-Eckmann complex structure on $S^3 \times S^3$ does not have vanishing first Chern class.

There is also a lot of interesting questions in this direction. See [18]. However we shall stop the discussion at this point, since it is outside the framework of the thesis.

Our purpose in this chapter is to derive more information about $M$ and the differential-geometrical structure of its moduli space. Especially we would like to enlighten the deformation theory for $M$ which for the present does not seem to be satisfactory.

At this stage, we have at our disposal only the Hodge diamond of $M$, a complex structure $J$ with $c_1(J) = 0$ and the compactness. Here are some direct consequences of these facts.

If we look at the Hodge diamond, we can see at first sight that $M$ is not a Kähler manifold. Indeed, the inequalities

$$0 \leq b_r \leq \sum_{p+q=r} h^{p,q},$$

which hold for any compact Hermitian manifold [19] imply that $b_2 = 0$. Note that in the Kähler case the second inequality is actually an equality which follows from the Hodge decomposition of Kähler manifolds.

Another natural question concerns the relationship between $h^{2,1}$ and $n$ - the number of copies of $S^3 \times S^3$ in the connected sum. For any compact complex manifold the Euler characteristic can be calculated by

$$\chi(M) = \sum_{r=0}^{\dim M} (-1)^r b_r = \sum_{p,q=0}^{\dim M/2} (-1)^{p+q} h^{p,q},$$

where $b_r$ is the $r$-th Betti number and $h^{p,q}$ is the respective Hodge number. The second equality is well-known for Kähler manifolds for the same reason we pointed out above - the particular Hodge decomposition of such manifolds. In the general Hermitian case this formula can be obtained by considering the Frölicher spectral sequences [19] which relate the cohomology groups of Dolbeault as invariants of the complex structure and the cohomology groups of De Rham as topological invariants. It can also be obtained by the Atiyah-Singer index theorem. For $M$ it gives

$$\chi = -2h^{2,1}.$$

See the Hodge diamond. On the other hand, under taking a connected sum of two even-dimensional manifolds $N_1$ and $N_2$, the Euler characteristic behaves as follows:

$$\chi(N_1 \# N_2) = \chi(N_1) + \chi(N_2) - 2.$$

See [3]. Thus

$$\chi(\#_n S^3 \times S^3) = \chi(\#_{n-1} S^3 \times S^3) - 2 = \dots = -2(n-1),$$

since

$$\chi(S^3 \times S^3) = \chi(S^3)\chi(S^3) = 0.$$

Therefore

$$h^{2,1} = n - 1.$$

Further we note that

$$H^1(M, \Theta) = H^1(M, \Omega^2) = H_{\bar\partial}^{2,1}(M),$$

where $\Theta$ is the sheaf of germs of the holomorphic vector fields over M, $\Omega^p$ - the sheaf of the holomorphic p-forms, and we have used the triviality of the canonical bundle which implies $\Theta \cong \Omega^2$. Moreover,

$$H^2(M, \Theta) = H^2(M, \Omega^2) = H_{\bar\partial}^{2,2}(M) = 0.$$

The spaces $H^1(\Theta)$ and $H^2(\Theta)$ play an important role in the theory of deformations of complex structures advanced by Kodaira and Spencer [34, 35]. A subsequent theorem of Kodaira, Nirenberg and Spencer [33] states that if a manifold has a complex structure J such that $H^2(\Theta) = 0$, then for any infinitesimal deformation I of J there exists an actual deformation of J which infinitesimally coincides with I. Therefore the space $H^2(\Theta)$ may be called obstruction space for the existence of infinitesimal deformations of the complex structure. As we saw above, it vanishes for $M$, so we can apply directly the Kodaira-Nirenberg-Spencer theorem [33], from which we conclude that the local moduli space of $M$ is smooth, that is, the first order deformations are unobstructed.

It is worthwhile to note that the K3 surfaces provided one of the earliest examples which illustrate the Kodaira-Spencer theory. Their moduli space is smooth of (real) dimension 40. As we pointed out in the first chapter the K3 surfaces admit a non-trivial Kähler-Einstein-Calabi-Yau metric. The Calabi-Yau manifolds are another recent example for the theory of deformations of complex structures. G.Tian [55] proved that the local moduli space of a Calabi-Yau manifold is smooth of dimension $\dim H^1(\Theta) = \dim H^1(\Omega^{m-1})$. Since for such manifolds $\Theta \cong \Omega^{m-1}$, the obstruction space $H^2(\Theta)$ is $H^2(\Omega^{m-1})$. The latter one need not be zero for Kähler manifolds. Indeed, for Kähler three-folds $H^2(\Omega^2)$ is never zero in the contrast to our situation on $\#_n S^3 \times S^3$.

Trying to trace some analogy between $M$ and a K3 surface we could say that both possess a nice moduli space (for $M$ at least locally). However, as is seen from Tian's proof or from the work of A.Todorov [57], the Calabi-Yau metric provides an important tool for the investigation of the moduli space. Moreover, on K3 fixing a complex structure and a cohomology class, namely the class of the Kähler form, determines uniquely the Calabi-Yau metric. Hence, the moduli space of Kähler-Einstein metrics on K3 is of dimension 57 ([4, 36]).

Now it is an open problem to have a Calabi-Yau substitute for non-Kähler manifolds. On the other hand there is the notion of stability which could suggest one such candidate. Indeed, Uhlenbeck and Yau [60] proved the existence of Hermitian-Einstein metrics in stable bundles over compact Kähler manifolds. The theorem of Uhlenbeck and Yau [60] was generalized by Li and Yau [38] for non-Kähler manifolds. The Kähler condition is replaced by the Gauduchon condition (for the definitions see section 2.3) which holds for a large class of Hermitian metrics. Therefore our first aim will be to prove the stability of the tangent bundle of $M$.

In section 2.4 we prove that there are no non-trivial line bundles on $M$. This enables us to deduce that its holomorphic tangent bundle is stable with respect to any Gauduchon metric. This is one of the main results in this chapter. Then we are in position to apply the theorem of Li and Yau [38], from which we conclude that there is an Hermitian-Einstein metric on the connected sum $\#_n S^3 \times S^3$. As far as is known to the author, the only (but very important) application of the Li-Yau theorem can be found in [39], where Yau et al. give a short proof of a famous theorem of Bogomolov.

Our basic hypothesis is that the Hermitian-Einstein metric and the Gauduchon metric coincide. Then, fixing the complex structure and the volume, we consider the deformations of the Hermitian-Einstein-Calabi-Yau-Li-Gauduchon metric. However, for this purpose we need to impose an additional condition which can be a substitute of the Kähler condition. We begin a search for different types of metrics which being Hermitian-Einstein admit no (essential) Hermitian deformations, that is, which are rigid for such manifolds. It is natural to suppose in the first place that such conditions are to be found in the terms of the torsion. We also seek the place of $M$ in the Gray-Hervella classification [25] of almost Hermitian manifolds. Further we make some notes on the deformations of both the metric and the complex structure. Especially we show that on any Hermitian non-Kähler manifold with $H^1(\mathcal{O}) = 0$ there are no non-zero anti-symmetric deformations of the complex structure. In section 2.2 we collect some elements of the Hermitian non-Kählerian geometry, some of which seem to be little known within the mathematical community. For completeness of the exposition, in section 2.3 we also recall the definition of stability, Hermitian-Einstein metrics, Gauduchon's condition, Li-Yau's theorem.

## 2.2 Elements of Hermitian non-Kählerian geometry

Let $M$ be a compact complex manifold of complex dimension $m \geq 2$, $\mathcal{T}$ - its tangent bundle and let $J : \mathcal{T} \longrightarrow \mathcal{T}$ be the almost complex structure induced by the complex structure in the standard way [62]. A Riemannian metric $g$ is called Hermitian if at each point $x \in M$

$$g(X, Y) = g(JX, JY)$$

for all $X, Y \in \mathcal{T}_x$. The fundamental or Kähler form $F$ of $g$ is given by

$$F(X,Y) = -g(X, JY).$$

A differential operator $D : \Gamma(\mathcal{T}) \longrightarrow \Gamma(\mathcal{T}^* \otimes \mathcal{T})$ which satisfies

$$D_X(fY) = Xf.Y + fD_XY$$

for every $X, Y \in \Gamma(\mathcal{T})$ and all $f \in C^\infty(M)$, is said to be a connection. It can be extended to higher tensor powers of the tangent and cotangent bundles in a natural way.

There are two tensors associated to $D$. The torsion $T$ of $D$ is a section in $\Lambda^2\mathcal{T}^* \otimes \mathcal{T}$,

$$T(X,Y) = D_XY - D_YX - [X,Y] \tag{2.1}$$

for $X, Y \in \Gamma(\mathcal{T})$. The curvature $R$ of $D$ is defined by

$$R(X,Y)Z = [D_Y, D_X]Z - D_{[Y,X]}Z$$

for all smooth vector fields $X, Y, Z$. With this sign convention the sphere $S^2$ has $R_{1212} > 0$.

All tensors can be extended to the complexification of $\mathcal{T}$:

$$\mathcal{T} \otimes C = T' \oplus T''$$

and to the tensor powers of $\mathcal{T}$ and $\mathcal{T}^*$. The tangent bundle $\mathcal{T}$ and the holomorphic tangent bundle $T'$ are identified. Since the related material is well-known (see e.g. [62]) we shall not present the details here.

A connection is said to

a) preserve the metric $g$, or to be a metric connection, or to be compatible with $g$, if $Dg = 0$, that is,

$$(Dg)(X,Y,Z) = Xg(Y,Z) - g(D_XY, Z) - g(Y, D_XZ) = 0;$$

b) preserve the complex structure $J$ or to be compatible with $J$ if $DJ = 0$, that is,

$$D_X(JY) = JD_XY.$$

There are two connections canonically associated to a given Hermitian metric $g$. According to A.Lichnerowicz [40], the first canonical connection $\nabla$ of $g$ is uniquely determined by the following defining properties:

$$\begin{aligned} \nabla g &= 0 \\ \nabla J &= 0 \\ T^{\nabla} &= 0, \end{aligned} \tag{2.2}$$

where $T^{\nabla}$ denotes the torsion of $\nabla$.

The Chern connection $D$ is the unique metric connection which preserves the complex structure and whose torsion $T$ is a vector valued (2,0) form, that is,

$$\begin{aligned} Dg &= 0 \\ DJ &= 0 \\ T(JX, Y) &= JT(X, Y). \end{aligned} \tag{2.3}$$

$D$ is also called the (standard) Hermitian connection, or in the terminology of A.Lichnerovicz "second canonical Hermitian connection" ([40]). For the Chern connection, in any local holomorphic frame, the corresponding connection forms are of type (1,0) with values in $End(\mathcal{T})$. In local complex coordinates adapted to the complex structure, $D$ has the well-known components:

$$\Gamma_{\alpha}{}^{\lambda}{}_{\beta} = g^{\lambda\bar{\mu}} \frac{\partial}{\partial z^{\alpha}} g_{\beta\bar{\mu}}.$$

A metric is Kähler if and only if its Levi-Civita and Chern connections coincide. From now on $D$ will denote the Chern connection of a Hermitian metric $g$.

Set

$$A(X, Y) = D_X Y - \nabla_X Y, \tag{2.4}$$

or in a local frame of $\mathcal{T}$

$$A_i{}^k{}_j = \Gamma_i{}^k{}_j - \tilde{\Gamma}_i{}^k{}_j,$$

where $\tilde{\Gamma}_i{}^k{}_j$ are the components of the Levi-Civita connection $\tilde{\nabla}$. The latin indices vary from 1 to $2m$, while the greek from 1 to $m$. In any local frame $e_j$ of $\mathcal{T}$ the components of the Levi-Civita connection are given by

$$\tilde{\Gamma}_i{}^k{}_j = \frac{1}{2}g^{km}(e_i g_{jm} + e_j g_{im} - e_m g_{ij}).$$

If $e_j = \partial/\partial x^j$, this is the well-known formula for the Christoffel symbols. However, we need to calculate them in a local complex frame adapted to the complex structure. If we take $e_j$ to be such a frame, then we shall have

$$\tilde{\Gamma}_\alpha{}^\lambda{}_\beta = \frac{1}{2}g^{\lambda\bar{\mu}}(\partial_\alpha g_{\beta\bar{\mu}} + \partial_\beta g_{\alpha\bar{\mu}}) = \frac{1}{2}(\Gamma_\alpha{}^\lambda{}_\beta + \Gamma_\beta{}^\lambda{}_\alpha).$$

Further

$$\begin{aligned}\tilde{\Gamma}_\alpha{}^{\bar{\lambda}}{}_{\bar{\beta}} &= \frac{1}{2}g^{\mu\bar{\lambda}}(\partial_\alpha g_{\mu\bar{\beta}} - \partial_\mu g_{\alpha\bar{\beta}})\\ &= \frac{1}{2}g^{\mu\bar{\lambda}}(\Gamma_\alpha{}^\sigma{}_\mu g_{\sigma\bar{\beta}} - \Gamma_\mu{}^\sigma{}_\alpha g_{\sigma\bar{\beta}})\\ &= \frac{1}{2}g^{\mu\bar{\lambda}}g_{\sigma\bar{\beta}}T_\alpha{}^\sigma{}_\mu = \frac{1}{2}T_{\alpha\bar{\beta}}{}^{\bar{\lambda}}\end{aligned}$$

and

$$\tilde{\Gamma}_\alpha{}^{\bar{\lambda}}{}_\beta = \tilde{\Gamma}_\alpha{}^\lambda{}_{\bar{\beta}} = 0.$$

Now it is easy to obtain that the components of $A$ with respect to a local complex frame adapted to the complex structure are expressed by

$$\begin{aligned} &A_\alpha{}^\lambda{}_\beta = \frac{1}{2}T_\alpha{}^\lambda{}_\beta, &&A_{\bar{\alpha}}{}^{\bar{\lambda}}{}_{\bar{\beta}} = \overline{A_\alpha{}^\lambda{}_\beta}\\ &A_\alpha{}^{\bar{\lambda}}{}_{\bar{\beta}} = A_{\bar{\beta}}{}^{\bar{\lambda}}{}_\alpha = \frac{1}{2}T^{\bar{\lambda}}{}_{\bar{\beta}\alpha}, &&A_{\bar{\alpha}}{}^\lambda{}_\beta = A_\beta{}^\lambda{}_{\bar{\alpha}} = \frac{1}{2}T^\lambda{}_{\beta\bar{\alpha}},\\ &A_\alpha{}^{\bar{\lambda}}{}_\beta = A_\alpha{}^\lambda{}_{\bar{\beta}} = 0. \end{aligned} \tag{2.5}$$

Indeed,

$$\begin{aligned}A_\alpha{}^\lambda{}_\beta &= \Gamma_\alpha{}^\lambda{}_\beta - \tilde{\Gamma}_\alpha{}^\lambda{}_\beta\\ &= \Gamma_\alpha{}^\lambda{}_\beta - \frac{1}{2}(\Gamma_\alpha{}^\lambda{}_\beta + \Gamma_\beta{}^\lambda{}_\alpha) = \frac{1}{2}T_\alpha{}^\lambda{}_\beta,\\ A_\alpha{}^{\bar{\lambda}}{}_{\bar{\beta}} &= -\tilde{\Gamma}_\alpha{}^{\bar{\lambda}}{}_{\bar{\beta}} = \frac{1}{2}T^{\bar{\lambda}}{}_{\bar{\beta}\alpha},\end{aligned}$$

etc. See also Gauduchon's paper [22] or his thesis [23]. The third condition in (2.3) is equivalent to $D'' = \bar\partial$ ([32]). Thus the curvature $R$ of $D$ is given by

$$R_{\alpha}{}^{\beta}{}_{\lambda\bar\mu} = \frac{\partial}{\partial \bar z^{\mu}} \Gamma_{\lambda}{}^{\beta}{}_{\alpha},$$

where

$$R_{\alpha}{}^{\beta}{}_{\lambda\bar\mu} \frac{\partial}{\partial \bar z^{\beta}} = R\left(\frac{\partial}{\partial z^{\lambda}}, \frac{\partial}{\partial \bar z^{\mu}}\right) \frac{\partial}{\partial z^{\alpha}}.$$

Taking the trace in (2.1) gives a form

$$\tau(Y) = trace\{X \longrightarrow T(X, Y)\}.$$

$\tau$ is called the **torsion 1-form.** By (2.1)

$$T_{\alpha}{}^{\lambda}{}_{\beta} = \Gamma_{\alpha}{}^{\lambda}{}_{\beta} - \Gamma_{\beta}{}^{\lambda}{}_{\alpha}.$$

The (1,0) form $\theta$ determined by

$$\theta_{\alpha} = T_{\lambda}{}^{\lambda}{}_{\alpha},$$

in a complex frame is called the **torsion (1,0) form.** The forms $\tau$ and $\theta$ are related by

$$\tau = \theta + \bar\theta. \tag{2.6}$$

Denote also $\Theta^{\alpha} = g^{\bar\lambda\alpha}\theta_{\bar\lambda}$.

**Definition 1.** *$M$ is said to be balanced if and only if $\theta = 0$.*

**Definition 2.** *$M$ is said to be semi-Kählerian if and only if $\tau = 0$.*

Definition 1 has been used by Michelson in [42] while Gauduchon has exploited Definition 2 [20, 22]. From (2.6) we see that $M$ is balanced if and only if it is semi-Kählerian. It is easy to prove([42]) that $M$ is balanced if and only if

$$P(F^{m-1}) = 0,$$

where $P$ is any of the operators $d, \partial, \bar\partial, d^{c}$. Under a conformal change $g' = e^{u}g$, the corresponding torsion forms are related by

$$T'_{\alpha}{}^{\lambda}{}_{\beta} = T_{\alpha}{}^{\lambda}{}_{\beta} + \frac{\partial u}{\partial z^{\alpha}}\delta^{\lambda}_{\beta} - \frac{\partial u}{\partial z^{\beta}}\delta^{\lambda}_{\alpha},$$

$$\theta' = \theta - (m-1)\partial u,$$

$$\tau' = \tau - (m-1)du. \tag{2.7}$$

(2.7) tells us that if

$$d\tau = 0,$$

then the class of $\tau$ in $H^1(M, R)$ is a conformal invariant.

Now take a trace in the curvature tensor. In contrast to the Kähler case we have three different Ricci tensors.

The second Ricci form $r$ can be obtained in the following way. Fix tangent vectors $Z, V$. Define $\dot{R}(Z, V)$ by

$$(\dot{R}(Z, V))(X, Y) = g(R(X, Y)Z, JV).$$

Hence for $X = \frac{\partial}{\partial z^\lambda}, Y = \frac{\partial}{\partial \bar{z}^\mu}, Z = \frac{\partial}{\partial z^\alpha}, V = \frac{\partial}{\partial \bar{z}^\gamma}$,

$$(\dot{R}_{\alpha\bar{\gamma}})_{\lambda\mu} = -ig_{\beta\bar{\gamma}}R_\alpha{}^\beta{}_{\lambda\mu}.$$

The curvature operator $R$ is by definition

$$R(\phi)(Z, V) = (\dot{R}(Z, V), \phi)_g.$$

$(,)_g$ is the inner product of forms of one and the same type, in which we omit $\det(g)$. In the case when $\phi$ is equal to the fundamental form $F$ of $g$ we have

$$((\dot{R}_{\alpha\bar{\gamma}})_{\lambda\mu}, \phi_{\lambda\mu})_g = g^{\lambda\tau}g^{\sigma\mu}(\dot{R}_{\alpha\bar{\gamma}})_{\lambda\mu}\phi_{\sigma\tau}$$

$$= g^{\lambda\tau}g^{\sigma\mu}(-i)R_{\alpha\bar{\gamma}\lambda\mu}ig_{\sigma\tau} = g^{\lambda\mu}R_{\alpha\gamma\lambda\mu}$$

$$= R_{\alpha\gamma\lambda}{}^\lambda = r_{\alpha\gamma},$$

that is, $r = R(F)$. The form $\rho = R^T(F)$-the transpose curvature operator, has components

$$\rho_{\lambda\mu} = R_\alpha{}^\alpha{}_{\lambda\mu}.$$

It is called the first Ricci form and $\frac{1}{2\pi}\rho$ represents the first Chern class of $M$. In fact,

$$\rho_{\lambda\mu} = \frac{\partial^2}{\partial z^\lambda \partial \bar{z}^\mu} \log\det(g). \tag{2.8}$$

The common trace

$$u = tr_g r = tr_g \rho$$

is the first scalar curvature of $g$, and

$$v = tr_g s$$

is the second scalar curvature, where $s$ is the third Ricci form determined by

$$s_{\lambda\mu} = R_\lambda{}^\alpha{}_{\alpha\mu}.$$

The first Bianchi identity

$$\mathcal{S}\{R(X,Y)Z\} = \mathcal{S}\{T(T(X,Y),Z) - (D_X T)(Y,Z)\},$$

where $\mathcal{S}$ means symmetrization of $X, Y, Z$ reads

$$R_\alpha{}^\beta{}_{\lambda\mu} - R_\lambda{}^\beta{}_{\alpha\mu} = -D_\mu T_\alpha{}^\beta{}_\lambda$$

Hence, by contraction

$$u - v = -D^\lambda \theta_\lambda. \tag{2.9}$$

We conclude this survey of the Hermitian non-Kählerian geometry by noting that one can consider the (2,1) torsion $\mathbf{T}$ with components

$$T_{\alpha\bar{\gamma}\beta} = g_{\lambda\bar{\gamma}} T_\alpha{}^\lambda{}_\beta$$

for which

$$\partial F = i\mathbf{T}/2. \tag{2.10}$$

## 2.3 Gauduchon metrics, stability and Li-Yau's theorem

To study an Hermitian manifold $M$ it is always useful to pick a metric with some special properties. Such metrics could be Kähler , balanced, Einstein,etc.. However, to admit a special metric the Hermitian manifold $M$ must satisfy some conditions, generally of a topological nature. First of all, one seeks a metric within a given conformal class. For instance, to obtain a balanced metric , i.e. $\partial(F^{m-1}) = 0$, in this way, in general is impossible. In many cases such metrics simply do not exist. What one can always achieve is due to the following result of Gauduchon [21, 20]:

**Theorem.** *Given any Hermitian metric on a compact complex manifold of dimension at least* 2, *there is a conformal metric unique up to homothety, such that its fundamental form satisfies*

$$\partial\bar{\partial}(F^{m-1}) = 0. \tag{2.11}$$

(2.11) is also equivalent to $\delta\theta = 0$ [20]. We shall call *Gauduchon metric* a metric for which the condition (2.11) holds. In his own terminology such metrics are said to be *standard* or of *null eccentricity*. In fact, there are many of them - one within each conformal class.

N.Hitchin observed in [28] that the Gauduchon metrics enable us to extend the notion of stability to holomorphic bundles on an arbitrary Hermitian manifold $M$. Namely, if $L$ is a holomorphic line bundle on $M$, its degree with respect to a given Gauduchon metric $F$ is

$$\deg(L) = \deg(L, F) = \frac{i}{2\pi}\int_M l \wedge F^{m-1},$$

where $l$ is the curvature of any Hermitian connection on $L$ compatible with $\bar{\partial}_L$, or more generally for a torsion-free coherent sheaf $\mathcal{S}$ on $M$

$$\deg(\mathcal{S}) = \int_M c_1(\mathcal{S}) \wedge F^{m-1}$$

is well-defined for the Gauduchon condition since any two first Chern forms differ by a $\partial\bar{\partial}$ - exact form. If $c_1 = 0$ then the degree vanishes.

Denote

$$\mu(\mathcal{S}) = \deg(\mathcal{S})/\mathrm{rank}(\mathcal{S}).$$

**Definition.** *$\mathcal{S}$ is called stable iff*

$$\mu(\mathcal{S}') < \mu(\mathcal{S})$$

*for any torsion-free subsheaf $\mathcal{S}'$ of $\mathcal{S}$.*

On the other hand there is the notion of Hermitian-Einstein metric. Let $(E, h)$ be a holomorphic vector bundle over $(M, g)$, where $h$ is an Hermitian metric in $E \longrightarrow M$ and $g$ is an Hermitian metric on $M$. Then the Chern connection $D$ of $h$ is said to be Hermitian-Einstein with respect to $g$ if and only if

$$g^{\lambda\mu} R(h)_{\alpha}{}^{\beta}{}_{\lambda\mu} = k.\delta^{\beta}_{\alpha}, \tag{2.12}$$

where $R(h)$ is the curvature of $D$. We shall also call $h$ an Hermitian-Einstein metric, when its Chern connection is Hermitian-Einstein. If $E = T$ - the tangent bundle of $M$, and if $g = h$, then (2.12) reads

$$r_{\alpha}{}^{\beta} = k.\delta^{\beta}_{\alpha}, \tag{2.13}$$

with $r$ - the second Ricci form of $g = h$. In the latter case if $g$ is Kähler, (2.13) is the well-known Kähler-Einstein condition.

In [41] Lübke proved if an indecomposible bundle $E$ over a Kähler manifold $M$ admits an Hermitian-Einstein metric, then it is stable. Later Uhlenbeck and Yau [60] proved the opposite statement. N.Hitchin suggested that the same relationship between stability and Hermitian-Einstein metrics should be also valid in the general Hermitian setting. Buchdahl [6] proved the theorem for surfaces and Li and Yau [38] generalized the work of Uhlenbeck - Yau to the non-Kähler case for all dimensions. Namely,

**The theorem of Li and Yau.**[38] *Let $M$ be a compact Hermitian manifold with a Gauduchon metric , and $E$ be a holomorphic vector bundle over $M$. Then $E$ is stable if and only if it admits an Hermitian-Einstein metric .*

## 2.4 Stability of the tangent bundle of $\#_n S^3 \times S^3$

**Proposition 2.1.** *There are no non-trivial line bundles on $M$.*

**Proof.** Consider the exponential exact sequence

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

and a part of the corresponding long exact sequence

$$H^1(Z) \to H^1(\mathcal{O}) \to H^1(\mathcal{O}^*) \xrightarrow{\delta} H^2(Z) \to H^2(\mathcal{O}) \to \ldots$$

We have

1.) $H^1(\mathcal{O}) \cong H^{0,1}_{\bar\partial}$ by the Dolbeault theorem. But $h^{0,1} = 0$. Therefore $H^1(\mathcal{O}) = 0$;

2.) since $M$ is simply connected $H^1(Z) = 0$;

3.) $H^2(\mathcal{O}) \cong H^{0,2}_{\bar\partial}$ and $h^{0,2} = 0$. Thus $H^2(\mathcal{O}) = 0$.

Then we have the exact sequence

$$0 \to H^1(\mathcal{O}^*) \xrightarrow{\delta} H^2(Z) \to 0.$$

Thus $\delta$ is an isomorphism and therefore

$$H^1(M, \mathcal{O}^*) \cong H^2(M, Z).$$

Further

$$\dim H^1(M, \mathcal{O}^*) = rg_Z H^2(M, Z) = rg_Z H_4(M, Z) = h^{2,2} = h^{1,1}$$

from the form of the Hodge diamond in [47]. Hence

$$H^1(M, \mathcal{O}^*) = 0 \quad \text{if and only if} \quad h^{1,1} = 0.$$

The latter holds for $M$.

From this proposition we deduce that there are no subvarieties of $M$ of codimension 1, that is, no divisors.

**Proposition 2.2.** *The (holomorphic) tangent bundle $\mathcal{T}$ of $M$ is stable with respect to any Gauduchon metric.*

**Proof.** By Proposition (7.6)(b') in [32],p.169, (also valid in the Hermitian case) it is sufficient to check the stability condition only for such subsheaves $S$ of $\mathcal{O}(\mathcal{T})$ for which the quotient sheaf $Q = \mathcal{O}(\mathcal{T})/S$ is torsion-free.

Let $S$ be a rank 1 - subsheaf of $\mathcal{O}(\mathcal{T})$ with $Q$ - torsion-free. Since $\mathcal{T}$ is a vector bundle, $\mathcal{O}(\mathcal{T})$ is a locally free sheaf and therefore torsion-free. Hence, $S$ is also torsion-free as subsheaf of $\mathcal{O}(\mathcal{T})$. Thus we have the following exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}(\mathcal{T}) \longrightarrow Q \longrightarrow 0.$$

Since $Q$ is torsion-free and $\mathcal{O}(\mathcal{T})$ is reflexive, from Lemma 1.1.16 in [44] it follows that $S$ is normal and being torsion-free, we get that $S$ is a reflexive rank 1 - sheaf. The latter means, equivalently, that $S$ is a line bundle. But from the **Proposition 2.1** we conclude that $S$ is the trivial line bundle. Its non-vanishing section is therefore a non-zero section of $\mathcal{O}(\mathcal{T}) = \Theta$. On the other hand, since the canonical bundle $K_M$ is trivial, that is,

$$K_M = \wedge^3 \mathcal{T}^* = \mathcal{O},$$

we have the pairing

$$\mathcal{T}^* \otimes \wedge^2 \mathcal{T}^* \longrightarrow \mathcal{O}$$

from which we obtain

$$\mathcal{T} \cong \wedge^2 \mathcal{T}^*$$

and

$$\Theta \cong \Omega^2.$$

So far, we have a non-zero holomorphic 2-form. This contradicts the fact that

$$\dim H^0(M, \Omega^2) = \dim H^{2\,0}(M) = h^{2\,0} = 0$$

which follows from the Dolbeault theorem and from the Hodge diamond. Therefore there are no rank 1 - subsheaves of $\Theta$ with torsion-free quotient.

Now suppose $E$ to be a rank 2 - subsheaf of $\Theta$ and let $F = \Theta/E$. We have the exact sequence

$$0 \longrightarrow E \longrightarrow \Theta \longrightarrow F \longrightarrow 0$$

and also a part of the dual long sequence

$$0 \longrightarrow F^* \longrightarrow \Omega^1 \longrightarrow \dots.$$

For an arbitrary coherent sheaf $A$ its dual $A^*$ is reflexive ([32], Proposition 5.18,p.160) and therefore $F^*$ is a rank 1 - reflexive sheaf, i.e. $F^*$ is a line bundle. Again from **Proposition 2.1** $F^*$ has to be the trivial line bundle and to have a non-vanishing section, which, from the inclusion $F' \longrightarrow \Omega^1$, provides a non-zero holomorphic 1-form. This is a contradiction since

$$\dim H^0(M, \Omega^1) = \dim H^{1,0}(M) = h^{1\,0} = 0.$$

Therefore there are no rank 2 - subsheaves of $\Theta$.

In this way we have proved the stability of the holomorphic tangent bundle of $\#_n S^3 \times S^3$ .

**Corollary 2.1.** There are no holomorphic subbundles of $\mathcal{T}$.

## 2.5 Application of Li-Yau's theorem to $\#_n S^3 \times S^3$ and some direct consequences

As we proved in the previous section, the tangent bundle of $M = \#_n S^3 \times S^3$ , $n \geq 103$, is stable with respect to any Gauduchon metric. Thus the theorem of Li and Yau applied to the tangent bundle implies that for any Gauduchon metric $g$ there exists an Hermitian-Einstein metric $h$, that is,

$$g^{\lambda\mu} R(h)_\alpha{}^\beta{}_{\lambda\mu} = k.\delta_\alpha^\beta, \tag{2.14}$$

where $R(h)$ is the curvature of the Chern connection, determined by $h$. By a conformal change of $h$, we can always make the function $k$ to be a constant [32].

By definition, up to some "immaterial" constants,

$$\deg(\mathcal{T}) = \int_M c_1(h) \wedge F^2 = \int_M tr_g c_1(h) \wedge F^3$$
$$= \int_M g^{\lambda\mu} R(h)_\alpha{}^\alpha{}_{\lambda\mu} dV_g = 3kVol(M)$$

for (2.14). Since $c_1(M) = 0$ the degree of the tangent bundle must be zero. Hence $k = 0$, and therefore

$$g^{\lambda\mu} R(h)_\alpha{}^\beta{}_{\lambda\mu} = 0. \tag{2.15}$$

Till this point the Gauduchon condition has been used only to have a definition of the degree which makes sense. Any Hermitian metric $g_1$ can be written as

$$g_1 = \varphi g, \tag{2.16}$$

where the smooth function $\varphi > 0$ is uniquely determined and $g$ is the respective Gauduchon metric in the conformal class of $g_1$ [21]. And vice-versa, any Gauduchon metric can be obtained from some Hermitian metric by (2.16). Hence, inserting (2.16) into (2.15) gives

$$g_1^{\lambda\bar{\mu}} R(h)_{\alpha}{}^{\beta}{}_{\lambda\bar{\mu}} = 0.$$

Since the tangent bundle is stable with respect to any Gauduchon metric, we see from the above equation and from the Li-Yau theorem that any Hermitian metric $g_1$ determines a unique Hermitian metric $h$ which is Hermitian-Einstein with respect to $g_1$. Of course, $h$ is Hermitian-Einstein with respect to any Hermitian metric in the conformal equivalence class of $g_1$. This is not a surprise since the Hermitian-Einstein condition is not "differential" with respect to the Gauduchon metric. Later on we shall use this freedom to impose on $g$ different conditions, some of which will be compatible with the Gauduchon condition.

Now let $\rho(h)$ be the first Ricci tensor of $h$. From (2.15) we obtain

$$g^{\lambda\bar{\mu}} \rho(h)_{\lambda\bar{\mu}} = 0. \qquad (2.17)$$

But $\rho(h)$ is given by (2.8). Therefore from (2.17) we conclude that

$$L(\log \det(h)) = 0, \qquad (2.18)$$

where

$$L = g^{\lambda\bar{\mu}} \frac{\partial^2}{\partial z^{\lambda} \partial \bar{z}^{\mu}}.$$

$L$ is an elliptic operator such that $L(1) = 0$. Hence (2.18) and the maximum principle of E.Hopf imply that

$$\det(h) = c = constant. \qquad (2.19)$$

The last remark is in fact a tautology since we look for $U(3)$ connection and the first Chern class vanishes.

Then from (2.8) and (2.19) we also get

$$\rho(h) = 0. \qquad (2.20)$$

As we pointed out in the Introduction, it is an open problem to have a substitute of the Calabi-Yau metric for non-Kähler manifolds. However, if we suppose that the Hermitian-Einstein metric $h$ and the Gauduchon metric $g$ coincide, this would be one possible candidate. Because it seems to be natural, henceforth we assume $g = h$ as our basic hypothesis. Then we shall look for some consequences of the existence of such a metric. The situation is similar to that on the K3 surfaces considered in the first chapter where we also had at our disposal two metrics: the Eguchi-Hanson and the Euclidean.

The hypothesis allows us to rewrite the Hermitian-Einstein condition as

$$r_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta}{}_{\lambda}{}^{\lambda} = 0. \tag{2.21}$$

Taking the trace in (2.21) we obtain

$$u = 0. \tag{2.22}$$

From (2.9) and (2.22) we also have

$$v = D^{\lambda}\theta_{\lambda}. \tag{2.23}$$

**Lemma 2.1.** *If $\varphi$ is a (1,0) form, then*

$$\delta\varphi = \delta'\varphi = -D^{\lambda}\varphi_{\lambda} + T_{\bar{\mu}}{}^{\mu\lambda}\varphi_{\lambda} \tag{2.24}$$

We postpone the proof of this lemma.

Since the metric is Gauduchon $\delta\theta = 0$ and from (2.23) and (2.24) for $\varphi = \theta$, we obtain

$$v = |\theta|^2 \geq 0.$$

Denote by $Scal$ the scalar curvature of an arbitrary Gauduchon metric, i.e. the scalar curvature of its Levi-Civita connection. The formula (33) in [20],p.505, gives

$$u = \frac{1}{2}Scal + \frac{1}{4}|dF|^2.$$

Hence and from (2.22) the scalar curvature of $g$ must be negative. It can not be identically zero since in this case the above formula would imply that $M$ is Kähler .

**Proof of Lemma 2.1.** Let $\psi$ be a function. Then

$$< \delta'\varphi, \psi > = < \varphi, \partial\psi > = < \varphi_\alpha, \partial_{\bar\alpha}\psi >$$

$$= \int_M g^{\alpha\mu}\varphi_\alpha . \partial_{\bar\mu}\bar\psi . g2^{m} dx.$$

For (2.19) we shall omit $g2^3 dx$. We integrate by parts:

$$< \delta'\varphi, \psi > = -\int_M g^{\alpha\mu}\partial_{\bar\mu}\varphi_\alpha . \bar\psi - \int_M (\partial_\mu g^{\alpha\mu})\varphi_\alpha . \bar\psi$$

$$= -\int_M D^\alpha \varphi_\alpha . \bar\psi + \int_M g^{\alpha\tau} g^{\sigma\bar\mu} \frac{\partial g_{\sigma\bar\tau}}{\partial \bar z^\mu} . \varphi_\alpha . \bar\psi$$

$$= -\int_M D^\lambda \varphi_\lambda . \bar\psi + \int_M T_{\bar\mu}{}^{\bar\mu\lambda}\varphi_\lambda . \bar\psi$$

since

$$g^{\sigma\bar\mu}\frac{\partial g_{\sigma\bar\tau}}{\partial \bar z^\mu} = \Gamma_{\bar\mu}{}^{\bar\mu}{}_{\bar\tau} = T_{\bar\mu}{}^{\bar\mu}{}_{\bar\tau}$$

for $\Gamma_{\bar\tau}{}^{\bar\mu}{}_{\bar\mu} = 0$ which follows from $\det(g) = const$.

In the next section we shall need to express $\delta F$ in the terms of the torsion 1-form $\tau$ and the torsion (1,0) form $\theta$. We shall make the calculations here.

**Lemma 2.2.** *The following formulae hold*

$$(\delta'' F)_\alpha = i\theta_\alpha, \qquad (\delta' F)_{\bar\alpha} = -i\theta_{\bar\alpha}.$$

**Proof.** Let $\phi$ be a (0,1) form. Then

$$< (\delta'' F)_\alpha, \phi_{\bar\alpha} > = < F_{\alpha\bar\beta}, \partial_\beta \phi_\alpha >$$

$$= \int_M i g_{\alpha\bar\beta} g^{\alpha\lambda} g^{\mu\beta}\partial_\mu \phi_{\bar\lambda} = i\int_M g^{\alpha\lambda}\partial_\alpha\phi_{\bar\lambda}.$$

Integrate by parts:

$$< (\delta'' F)_\alpha, \phi_{\bar\alpha} > = -i\int_M \partial_\alpha g^{\alpha\lambda} . \phi_{\bar\lambda} = i\int_M \theta_\alpha g^{\alpha\lambda}\phi_{\bar\lambda} = < i\theta_\alpha, \phi_{\bar\alpha} >$$

for the same reasons as in the proof of **Lemma 2.1.** Therefore

$$(\delta'' F)_\alpha = i\theta_\alpha$$

and

$$(\delta' F)_{\bar\alpha} = -i\theta_{\bar\alpha}.$$

We also have

$$\tau(X) = -\delta F(JX)$$

$$\tau(JX) = \delta F(X).$$

Indeed, in local complex coordinates (2.6) reads

$$\tau_{\alpha} = \theta_{\alpha}, \qquad \tau_{\bar\alpha} = \theta_{\bar\alpha}.$$

Then

$$\tau_{\alpha} = -\delta F(J\partial_{\alpha}) = -i(\delta^{c} F)_{\alpha} = -i.i\theta_{\alpha} = \theta_{\alpha}.$$

Since in (2.5) we have already calculated the components of $A$, we could obtain these formulae as well as (2.24) without using the condition $\det(g) = const$ in the same way as in the Appendix of [22]. However, the calculation is lengthy and for this we chose the above way in order not to increase the volume of the thesis.

## 2.6 Some notes on the classification of almost Hermitian manifolds by Gray and Hervella

According to the Gray-Hervella classification [25] there are sixteen classes of almost Hermitian manifolds. They can be divided in three groups. The first group contains the following neighbouring types : Kähler, Nearly Kähler, Almost Kähler, Quasi-Kähler manifolds. If such a manifold $M$ is Hermitian, then $M$ must be Kähler . Indeed, with the same notations as in [25] we have

1k) If $M$ is a Nearly Kähler manifold, that is, $M \in W_1$, then by the linear relations among the invariants which characterize each of the sixteen classes ([25]) it follows that

$$\|\nabla F\|^2 = \frac{1}{16}\|S\|^2 = 0.$$

Here $S$ denotes the Nijenhuis tensor whose vanishing determines the class of the Hermitian manifolds. Hence $F$ is parallel with respect to the Levi-Civita connection $\nabla$ and therefore $M$ is Kähler.

2k) If $M$ is Almost Kähler, that is, $M \in W_2$, by the same relations

$$\|\nabla F\|^2 = \frac{1}{4}\|S\|^2 = 0$$

and $M$ is Kähler.

3k) If $M$ is Quasi-Kähler then by the defining condition

$$\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z) = 0$$

([25],p.41) and if it is in the class $H = W_3 \oplus W_4$ of Hermitian manifolds,

$$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0 \qquad (*)$$

([25],p.41). The last two equations imply

$$\nabla_X(F)(Y,Z) = 0,$$

that is, $M$ is Kähler .

The second group consists of almost Hermitian manifolds which admit a semi-Kählerian metric, i.e. which satisfy $\delta F = \tau = 0$. For this group we consider the following cases :

1s) By definition the class $W_3 = SK \cap H$ of all semi-Kählerian manifolds is Hermitian and therefore

$$W_3 \cap H = W_3$$

2s) For the class $W_1 \oplus W_2 \oplus W_3 = SK$ of all almost Hermitian semi-Kählerian manifolds the relation

$$(W_1 \oplus W_2 \oplus W_3) \cap H = SK \cap H = W_3$$

is also obvious.

3s) The class $W_1 \oplus W_3$ is defined by

$$\nabla_X(F)(X,Z) - \nabla_{JX}(F)(JX,Z) = 0$$

$$\delta F = 0.$$

But $H$ is determined by $(*)$. Thus the above relation is fulfilled for any Hermitian manifold and therefore

$$(W_1 \oplus W_3) \cap H = H \cap \{\delta F = 0\} = W_3.$$

4s) The class $W_2 \oplus W_3$ is defined by

$$\mathcal{S}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = 0$$

$$\delta F = 0.$$

Analogously to 3s) we get

$$(W_1 \oplus W_3) \cap H = W_3.$$

We conclude that the intersection with $H$ of any manifold in this group is the class $W_3$ of all Hermitian semi-Kählerian manifolds.

The third group contains eight classes which are preserved under conformal changes of the metric. In what follows, the paper of Gray and Hervella [25] is again our fundamental reference.

1w) The class $W_1 \oplus W_3 \oplus W_4$ has the following definition

$$\nabla_X(F)(X, Z) - \nabla_{JX}(F)(JX, Z) = 0,$$

which is a consequence of $(*)$. Therefore

$$(W_1 \oplus W_3 \oplus W_4) \cap H = H$$

2w) $W_2 \oplus W_3 \oplus W_4$ is defined by

$$\mathcal{S}\{\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z)\} = 0.$$

This relation follows from $(*)$ and hence

$$(W_2 \oplus W_3 \oplus W_4) \cap H = H$$

3w) It is obvious that

$$W \cap H = H$$

where $W$ is the class of the almost Hermitian manifolds.

4w) $W_4$ is always complex and therefore

$$W_4 \cap H = W_4.$$

It is a class which contains locally conformally Kähler manifolds and whose defining condition is

$$\nabla_X(F)(Y, Z) = \frac{1}{2(m-1)}[g(X, Y)\delta F(Z) - g(X, Z)\delta F(Y)$$

$$-g(X, JY)\delta F(JZ) + g(X, JZ)\delta F(JY)].$$

As we shall see below (**Lemma 2.4**) this is equivalent to

$$\partial F + \frac{1}{m-1} F \wedge \theta = 0.$$

5w) The defining condition for the class $W_1 \oplus W_2 \oplus W_4$ is

$$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)(JY, Z) = \frac{1}{(m-1)}[g(X, Y)\delta F(Z)$$

$$-g(X, Z)\delta F(Y) - g(X, JY)\delta F(JZ) + g(X, JZ)\delta F(JY)].$$

Hence and from $(*)$ and 4w) we obtain

$$(W_1 \oplus W_2 \oplus W_4) \cap H = W_4$$

6w) Of course

$$(W_3 \oplus W_4) \cap H = H.$$

7w) and 8w) The intersections of $W_1 \oplus W_4$ and $W_2 \oplus W_4$ with $H$ coincide with $W_4$. This can be obtained by the components of a tensor $\mu$ which is conformally invariant and characterizes the eight manifolds in this group. See **Lemma 2.3** and **Lemma 2.4** below.

The next proposition summarizes our deductions.

**Propositon 2.3.** *There are four classes of Hermitian manifolds of dimension* $\geq 3$. *Namely*

*1. Kähler manifolds* : $dF = 0$

*2. Semi-Kähler manifolds* : $\delta F = 0$

*3.* $W_4$ : $\partial F + \frac{1}{m-1} F \wedge \theta = 0$

*4. H - Hermitian manifolds.*

We note that the Gray-Hervella classification (as well as the above one) is up to some conditions which are imposed on the torsion, that is, on first derivatives of the metric. For instance, a metric is Kähler if and only if its torsion vanishes, a metric is semi-Kähler if and only if a part of the torsion vanishes, namely its trace with respect to the metric, which we called in section 2.2 the torsion 1 - form.

The above third group of almost Hermitian manifolds is characterized by a tensor $\mu$, which we shall calculate in a local complex frame adapted to the complex structure, assuming that the considered manifolds are Hermitian.

$\mu$ is a tensor field given by

$$g(\mu(X,Y),Z) = \nabla_X(F)(Y,Z) - \frac{1}{2(m-1)}[g(X,Y)\delta F(Z) - g(X,Z)\delta F(Y)$$

$$-g(X,JY)\delta F(JZ) + g(X,JZ)\delta F(JY)].$$

We are going to express $\mu$ in the terms of the torsion. For this purpose we note that

$$\nabla_X(F)(Y,Z) = XF(Y,Z) - F(\nabla_X,Z) - F(Y,\nabla_X Z)$$

$$= -Xg(Y,JZ) + g(\nabla_X,JZ) + g(Y,J\nabla_X Z)$$

$$= -Xg(Y,JZ) + g(D_X Y - A(X,Y),JZ) + g(Y,J(D_X Z - A(X,Z))$$

for (2.4). Then

$$\nabla_X(F)(Y,Z) = -Xg(Y,JZ) + g(D_X Y,JZ) + g(Y,JD_X Z)$$

$$-g(A(X,Y),JZ) - g(Y,JA(X,Z))$$

$$= -g(A(X,Y),JZ) - g(Y,JA(X,Z))$$

since by definition the Chern connection $D$ preserves both $g$ and $J$ (see (2.3)). Hence, and from the formulae at the end of the previous section we deduce that the tensor

$\mu$ can be expressed in the following form

$$g(\mu(X,Y),Z) = -g(A(X,Y),JZ) - g(Y,JA(X,Z))$$

$$-[g(X,Y)\tau(JZ) - g(X,Z)\tau(JY) + g(X,JY)\tau(Z) - g(X,JZ)\tau(Y)]/(2(m-1)) \quad (2.25)$$

Now we begin the evaluating of (2.25) in a local frame.

1. Let $X = \partial_\alpha (= \frac{\partial}{\partial z^\alpha}), Y = \partial_\beta, Z = \partial_\gamma$. Write

$$\mu(X,Y) = \mu_\alpha{}^\sigma{}_\beta \partial_\sigma + \mu_\alpha{}^{\bar\sigma}{}_\beta \partial_{\bar\sigma}.$$

From (2.5) and (2.25) we get

$$g_{\sigma\bar\gamma}\,\mu_\alpha{}^\sigma{}_\beta = -g(A_\alpha{}^\sigma{}_\beta \partial_\sigma + A_\alpha{}^{\bar\sigma}{}_\beta \partial_{\bar\sigma}, -i\partial_{\bar\gamma})$$

$$-g(\partial_\beta, J(A_\alpha{}^\sigma{}_{\bar\gamma} \partial_\sigma + A_\alpha{}^{\bar\sigma}{}_{\bar\gamma} \partial_{\bar\sigma}))$$

$$= ig_{\sigma\bar\gamma} A_\alpha{}^\sigma{}_\beta + ig_{\beta\bar\sigma} A_\alpha{}^{\bar\sigma}{}_{\bar\gamma}$$

$$= ig_{\sigma\bar\gamma} T_\alpha{}^\sigma{}_\beta/2 + ig_{\beta\bar\sigma} T^{\bar\sigma}{}_{\bar\gamma\alpha}/2 = i(T_{\alpha\bar\gamma\beta} + T_{\beta\bar\gamma\alpha})/2 = 0.$$

That is,

$$\mu_\alpha{}^\sigma{}_\beta = 0.$$

2. If $X = \partial_\alpha, Y = \partial_\beta, Z = \partial_\gamma$, then it is easy to see that

$$\mu_\alpha{}^{\bar\sigma}{}_\beta = 0.$$

3. Put $X = \partial_\alpha, Y = \partial_{\bar\beta}, Z = \partial_\gamma$. Analogously to 1. we get

$$\mu_\alpha{}^{\bar\sigma}{}_{\bar\beta} = 0.$$

4. Now let $X = \partial_{\bar\alpha}, Y = \partial_\beta, Z = \partial_\gamma$. Then

$$g(\mu(X,Y),Z) = g_{\gamma\bar\sigma}\mu_{\bar\alpha}{}^{\bar\sigma}{}_\beta = -ig_{\gamma\bar\sigma} A_{\bar\alpha}{}^{\bar\sigma}{}_\beta + ig_{\beta\bar\sigma} A_{\bar\alpha}{}^{\bar\sigma}{}_\gamma$$

$$+i[g_{\beta\bar\alpha}\theta_\gamma - g_{\gamma\bar\alpha}\theta_\beta + g_{\beta\bar\alpha}\theta_\gamma - g_{\gamma\bar\alpha}\theta_\beta]/(2(m-1))$$

$$= -ig_{\gamma\bar\sigma} T^{\bar\sigma}{}_{\bar\alpha\beta}/2 + ig_{\beta\bar\sigma} T^{\bar\sigma}{}_{\bar\alpha\gamma}/2 - i(g_{\beta\bar\alpha}\theta_\gamma - g_{\gamma\bar\alpha}\theta_\beta)/(m-1)$$

$$= i[-T_{\gamma\bar\alpha\beta} + T_{\beta\bar\alpha\gamma}]/2 - i(g_{\beta\bar\alpha}\theta_\gamma - g_{\gamma\bar\alpha}\theta_\beta)/(m-1)$$

$$= i[T_{\beta\bar\alpha\gamma} - \frac{1}{m-1}(g_{\beta\bar\alpha}\theta_\gamma - g_{\gamma\bar\alpha}\theta_\beta)].$$

that is,

$$\mu_{\bar\alpha}{}^{\sigma}{}_{\beta} = i[T_{\beta\bar\alpha}{}^{\sigma} - \frac{1}{m-1}(\Theta^\sigma g_{\beta\bar\alpha} - \theta_\beta\delta^\sigma_{\bar\alpha})].$$

All other components are complex conjugate to the ones just calculated. We have proved

**Lemma 2.3.** *In a local complex frame adapted to the complex structure, the only nonvanishing components of the tensor $\mu$ for any Hermitian manifold are given by*

$$\mu_{\bar\alpha}{}^{\sigma}{}_{\beta} = i[T_{\beta\bar\alpha}{}^{\sigma} - \frac{1}{m-1}(\Theta^\sigma g_{\beta\bar\alpha} - \theta_\beta\delta^\sigma_{\bar\alpha})] = \overline{\mu_{\alpha}{}^{\bar\sigma}{}_{\bar\beta}}. \tag{2.26}$$

Now we present the proof which we promised in 4w), 7w) and 8w). Namely,

**Lemma 2.4.** *The class $W_4$ is determined by*

$$\partial F + \frac{1}{m-1}\theta \wedge F = 0,$$

*which is a conformally invariant condition. Moreover, the following relations hold:*

$$(W_1 \oplus W_4) \cap H = W_4,$$

$$(W_2 \oplus W_4) \cap H = W_4.$$

**Proof.** Suppose that $M$ is in $W_4$. By the definitions of $\mu$ and $W_4$ this is equivalent to

$$\mu(X, Y) = 0$$

for any smooth vector fields $X$ and $Y$. Hence, it is easy to obtain by **Lemma 2.3** that $M \in W_4$ if and only if

$$T_{\bar\alpha}{}^{\lambda}{}_{\beta} = \frac{1}{m-1}(\theta_\beta\delta^\lambda_{\bar\alpha} - \theta_{\bar\alpha}\delta^\lambda_\beta) \tag{2.27}$$

For (2.10), (2.27) can be rewritten in the form

$$\partial F = -\frac{1}{m-1}\theta \wedge F,$$

which appeared in **Proposition 2.3.** From the two formulae just before (2.7) we obtain easily that the equation (2.27) is conformally invariant.

Let us now explore the case when $M \in W_1 \oplus W_2$. This holds if and only if

$$\mu(X, X) = 0 \tag{2.28}$$

for all smooth vector fields $X$. See [25]. From (2.26) we get easily that (2.28) is again equivalent to (2.27).

Further, $M \in W_2 \oplus W_4$ if and only if

$$\mathcal{S} g(\mu(X, Y), Z) = 0 \tag{2.29}$$

for all smooth vector fields $X, Y, Z$ ([25]). Recall that $\mathcal{S}$ means symmetrization of $X, Y, Z$. We shall obtain a different form of the left-hand side of (2.29). By (2.25)

$$\mathcal{S} g(\mu(X, Y), Z) = -g(A(X, Y), JZ) - g(A(Y, Z), JX) - g(A(Z, X), JY)$$

$$+g(JY, A(X, Z)) + g(JX, A(Z, Y)) + g(JZ, A(Y, X))$$

$$-[g(X, Y)\tau(JZ) + g(Y, Z)\tau(JX) + g(Z, X)\tau(JY)$$

$$-g(X, Z)\tau(JY) - g(Z, Y)\tau(JX) - g(Y, X)\tau(JZ)$$

$$+g(X, JY)\tau(Z) + g(Y, JZ)\tau(X) + g(Z, JX)\tau(Y)$$

$$-g(X, JZ)\tau(Y) - g(Z, JY)\tau(X) - g(Y, JX)\tau(Z)]/(2(m-1))$$

$$= g(A(Y, X) - A(X, Y), JZ) + g(A(Z, Y) - A(Y, Z), JX) + g(A(X, Z) - A(Z, X), JY)$$

$$-[g(X, JY)\tau(Z) + g(Y, JZ)\tau(X) + g(Z, JX)\tau(Y)]/(m-1)$$

$$= \mathcal{S}[-g(T(X, Y), JZ) + F(X, Y)\tau(Z)/(m-1)]$$

since $F(X, Y) = -g(X, JY)$ and by (2.1) and (2.4) :

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

$$= \nabla_X Y + A(X, Y) - \nabla_Y X - A(Y, X) - [X, Y]$$

$$= T^{\nabla}(X, Y) + A(X, Y) - A(Y, X)$$

$$= A(X,Y) - A(Y,X)$$

since the Levi-Civita connection $\nabla$ has vanishing torsion. By (2.64) in the section 2.11

$$\mathcal{S}g(\mu(X,Y),Z) = 2dF(X,Y,Z) + \frac{2}{m-1}(F \wedge \tau)(X,Y,Z).$$

Hence, (2.29) is equivalent to

$$dF + \frac{1}{m-1}F \wedge \tau = 0. \tag{2.30}$$

This completes the proof of **Proposition 2.3.**

We would like to mention that the traceless tensor $dF + \frac{1}{m-1}F \wedge \tau$ has been introduced by P.Libermann and is called conformal torsion.

**Remark.** The condition

$$\mu(X,Y) - \mu(JX,JY) = 0 \tag{2.31}$$

must be fulfilled for any Hermitian manifold. (2.31) can be checked easily using the components of $\mu$ given by (2.26).

## 2.7 $\#_n S^3 \times S^3$ and its place in the the Gray-Hervella classification

According to the **Proposition 2.3** in section 2.6, there are four classes of Hermitian manifolds. As we noted in the Introduction $M = \#_n S^3 \times S^3$ is not a Kähler manifold. So we reject the first case.

Now suppose $M$ is a locally conformal Kähler manifold. Let $F$ be the corresponding Hermitian form. In a local chart $U_\alpha$ we have

$$F|_{U_\alpha} = e^{f_\alpha}\omega_\alpha,$$

$\omega_\alpha$ being the "locally Kähler" form, that is, $d\omega_\alpha = 0$. Then in $U_\alpha \cap U_\beta$

$$\omega_\beta = e^{f_\alpha - f_\beta}\omega_\alpha.$$

But

$$d\omega_\alpha = d\omega_\beta = 0.$$

Therefore $f_\alpha - f_\beta$ determines a Cech cocycle in $H^1(M, R)$. Since $b_1 = 0$, $f_\alpha - f_\beta$ must be exact, that is,

$$f_\alpha - f_\beta = c_\alpha - c_\beta$$

for some constants $c_\alpha$. Define a global function $g$ by

$$f_\alpha - c_\alpha = g|_{U_\alpha}$$

It is well-defined since $f_\alpha - c_\alpha = f_\beta - c_\beta$. Then

$$(e^{-g}F)|_{U_\alpha} = e^{-f_\alpha + c_\alpha} e^{f_\alpha}\omega_\alpha = e^{c_\alpha}\omega_\alpha$$

and

$$d(e^{-g}F)|_{U_\alpha} = d(e^{c_\alpha}\omega_\alpha) = e^{c_\alpha} d\omega_\alpha = 0.$$

Hence there is a Kähler metric on $M$, which is impossible since $b_2 = 0$. Therefore we proved the following

**Lemma 2.5.** *$M$ is not a locally conformal Kähler manifold.*

The next proposition is a straightforward consequence of that in the previous section.

**Proposition 2.4.** *Let $M = \#_n S^3 \times S^3$ , $n \geq 103$, and $\theta$ be the torsion $(1,0)$ form of an Hermitian metric $g$ on $M$, given by $\theta_\alpha = T_\lambda{}^\lambda{}_\alpha$, where $T$ is the torsion of $g$. Then there are three possible cases :*

(1) *(M,g) is semi-Kähler :*

$$\theta = 0;$$

(2) *(M,g) has vanishing conformal torsion, that is,*

$$T_\alpha{}^\lambda{}_\beta = \frac{1}{2}(\theta_\beta \delta_\alpha^\lambda - \theta_\alpha \delta_\beta^\lambda)$$

*or equivalently*

$$\partial F = -\frac{1}{2}\theta \wedge F.$$

(3) *(M,g) is a general Hermitian manifold, i.e. with no condition imposed on the torsion,but not on its derivatives.*

The case (3) above permits us to impose conditions on the first derivatives of the torsion, that is, on the second derivatives of the metric. In the next section we shall consider two such conditions.

We also note that, despite the strict inclusions between the sixteen classes in Gray-Hervella classification [25], one and the same manifold $M$ could be in different classes but with different Hermitian metrics.

We call the cases (1), (2) and (3) respectively S, C and H. From **Lemma 2.5** we deduce that $\mathrm{C} = W_4 \setminus$ {locally conformal Kähler manifolds}.

In the Kähler case there are various topological consequences which the vanishing of the torsion implies. We used one of them to deduce that $M$ is not Kähler since $b_2 = 0$. In [42] Michelson proves a necessary and sufficient condition for existence of a balanced metric on a compact complex manifold. Michelson's criterion is of a topological nature. Its applying to $\#_n S^3 \times S^3$ gives the following lemma in which we shall use the same notations and terminology of [42], without recalling.

**Lemma 2.3.** *$M$ is not balanced if and only if there is a closed current $T \in \mathcal{E}_4'(M)$ such that $\pi_{2,2}T \neq 0$ is in the positive cone $\mathcal{P}_{2,2}$.*

**Proof.** Denote by $Z_p'$ the closed $p$-currents. By definition

$$Z_4' = H_4 + d\mathcal{E}_5',$$

where $H_4$ is the corresponding homology group of currents. A result of Federer and Fleming [16] states that there is an isomorphism

$$H_4 = H_4(M, C).$$

Since $b_4 = 0$,

$$H_4(M, C) = 0.$$

Hence

$$Z_4' = d\mathcal{E}_5'$$

and also

$$\pi_{2,2} Z_4' \cap \mathcal{P}_{2,2} = \pi_{2,2} d\mathcal{E}_5' \cap \mathcal{P}_{2,2}. \tag{2.32}$$

By the characterization theorem of Michelson [42] $M$ is balanced if and only if the right-hand side of (2.32) is 0. This completes the proof.

Thus if there is a generic example of a closed current $T \in \mathcal{E}_4'$ such that $\pi_{2,2} T \neq 0$ is in $\mathcal{P}_{2,2}$, then $M$ is not balanced=semi-Kähler.

## 2.8 Some conditions on $\#_n S^3 \times S^3$

We begin a search for different sorts of conditions which can be imposed in order to have a rigidity theorem which would suggest the existence of a canonical metric. Such a metric could be the Hermitian-Einstein metric, deduced in section 2.5 from the stability of the tangent bundle of $\#_n S^3 \times S^3$. However, we need an additional condition, which replaces the Kähler condition. Possible conditions are (1) and (2) in the **Proposition 2.4.** They involve only first derivatives of the metric. If $M$ is not in both S and C, $M$ will be a "general" Hermitian manifold without any conditions on the first derivatives of the metric. Thus, except the Hermitian-Einstein condition, we shall seek some natural conditions in terms of second derivatives.

To begin with, note that the torsion $T$ is a (2,0) vector-valued form, that is, $T \in \wedge^{2,0} \otimes \mathcal{T}$, where $\mathcal{T}$ is the holomorphic tangent bundle. Now if $T$ is holomorphic :

$$\partial_{\bar\mu} T_\alpha{}^\sigma{}_\lambda = 0, \tag{2.33}$$

that is,

$$D_{\bar\mu} T_\alpha{}^\sigma{}_\lambda = T_\alpha{}^\sigma{}_{\lambda|\bar\mu} = 0.$$

However, this cannot happen on $M$ as the following proposition asserts.

**Proposition 2.5.** *The torsion of any Hermitian-Einstein metric on $\#_n S^3 \times S^3$ is not holomorphic.*

**Proof.** Suppose that $T$ is holomorphic. Then from **Lemma 2.7** below it follows that

$$T_\alpha{}^\sigma{}_{\lambda|\mu} = 0,$$

that is, $T$ is parallel. (2.33) means that

$$T \in H^{2,0}_{\bar{\partial}}(M, \mathcal{T}) = H^0(M, \Omega^2(\mathcal{T})) = H^0(M, \Theta \otimes \mathcal{T})$$

since $K_M$ is trivial and therefore $\Omega^2 = \Theta$. In this way we see that $T$ determines a map

$$t : \mathcal{T} \longrightarrow \mathcal{T}^*.$$

and also

$$\det(t) : \wedge^3 \mathcal{T} \longrightarrow \wedge^3 \mathcal{T}^*.$$

But the canonical bundle $K_M = \wedge^3 \mathcal{T}^*$ is trivial. Hence, if the rank of $\det(t)$ is not maximal, the kernel of $t$ would be a non-trivial holomorphic subbundle of $\mathcal{T}$ since $T$ is parallel and therefore nowhere vanishing. However, according to the **Corollary 2.1** in the section 2.4, $\mathcal{T}$ does not have any non-trivial holomorphic subbundles. Therefore $\det(t)$ has maximal rank and in this case $t$ must be an isomorphism. This means that the holonomy group of $M$ is included in $O(3, C)$. Hence, since $M$ is an Hermitian manifold , it is the maximal compact subgroup of $SU(3) \cap O(3, C)$. Thus, the holonomy group is reduced to $SO(3)$ and the tangent bundle has the following decomposition :

$$\mathcal{T} = E \otimes C = E \otimes iE,$$

where $E$ is a real rank 3 - bundle. By the defining properties of the Chern classes

$$c_i(E \otimes C) = 0$$

if $i$ - odd. This is a contradiction since the Euler characteristic of $M$ is $-2(n-1) \leq -204 < 0$ and therefore $c_3(\mathcal{T}) = c_3(E \otimes C) \neq 0$. Conclusion:

$$\partial_\mu T_\sigma{}^\sigma{}_\lambda = 0$$

cannot happen.

We would like to note that the above proof holds for $n \geq 2$, but not for $n = 1$ if there is actually a complex structure with trivial canonical class on $S^3 \times S^3$. However, if $M$ is not balanced, we have a short proof. Namely, if the torsion is holomorphic, then by a contraction we can obtain that the torsion (1,0) form $\theta$ is also holomorphic and therefore it must be zero since $h^{1,0} = 0$. Hence, $M$ is balanced. This proof covers also the case $n = 1$.

**Lemma 2.7.** *If the torsion of an Hermitian-Einstein metric on $M$ is holomorphic, then it is parallel.*

**Proof.** Let

$$f = \sum T_\alpha{}^\sigma{}_\lambda \bar{T}_\alpha{}^\sigma{}_\lambda.$$

From (2.33)

$$\begin{aligned} L(f) = \sum f_{\mu\bar\mu} &= \sum (T_\alpha{}^\sigma{}_{\lambda|\mu} \bar{T}_\alpha{}^\sigma{}_\lambda + T_\alpha{}^\sigma{}_\lambda \bar{T}_\alpha{}^\sigma{}_{\lambda|\mu})_{|\bar\mu} \\ &= \sum (T_\alpha{}^\sigma{}_{\lambda|\mu\bar\mu} \bar{T}_\alpha{}^\sigma{}_\lambda + T_\alpha{}^\sigma{}_{\lambda|\mu} \bar{T}_\alpha{}^\sigma{}_{\lambda|\mu}). \end{aligned}$$

But

$$\begin{aligned} T_\alpha{}^\sigma{}_{\lambda|\nu\bar\gamma} - T_\alpha{}^\sigma{}_{\lambda|\bar\gamma\nu} &= \sum_\beta T_\beta{}^\sigma{}_\lambda R_{\alpha\beta\nu\bar\gamma} \\ &+ \sum_\beta T_\alpha{}^\sigma{}_\beta R_{\lambda\bar\beta\nu\bar\gamma} - \sum_\beta T_{\alpha\beta\lambda} R_\beta{}^\sigma{}_{\nu\bar\gamma}. \end{aligned}$$

Let $\mu = \nu = \gamma$ and sum:

$$\begin{aligned} T_\alpha{}^\sigma{}_{\lambda|\mu\bar\mu} - T_\alpha{}^\sigma{}_{\lambda|\bar\mu\mu} &= \sum_{\beta,\mu} (T_\beta{}^\sigma{}_\lambda R_{\alpha\beta\mu\bar\mu} \\ &+ T_\alpha{}^\sigma{}_\beta R_{\lambda\bar\beta\mu\bar\mu} - T_{\alpha\beta\lambda} R_\beta{}^\sigma{}_{\mu\bar\mu}) = 0 \end{aligned}$$

since $g$ is Hermitian-Einstein. Therefore

$$T_\alpha{}^\sigma{}_{\lambda|\mu\bar\mu} = T_\alpha{}^\sigma{}_{\lambda|\bar\mu\mu} = ((T_\alpha{}^\sigma{}_\lambda)_{|\bar\mu})_{|\mu} = 0$$

and

$$L(f) = \sum T_\alpha{}^\sigma{}_{\lambda|\mu} \bar{T}_\alpha{}^\sigma{}_{\lambda|\bar\mu} \geq 0. \tag{2.34}$$

By the maximum principle of E.Hopf (or by the Bochner lemma) it follows that

$$L(f) = 0.$$

Therefore from (2.34)

$$T_\alpha{}^\sigma{}_{\lambda|\mu} = 0.$$

In fact,

$$\partial_\mu T_\alpha{}^\sigma{}_\lambda = 0$$

if and only if

$$T_\alpha{}^\sigma{}_{\lambda|\mu} = 0.$$

For complex surfaces the Gauduchon condition is

$$\partial\bar{\partial} F = 0. \tag{2.35}$$

Since we have not essentially used the Gauduchon condition, (2.35) seems to be a nice substitute in higher dimensions. From (2.10) and (2.35) we obtain

$$\bar{\partial}\mathbf{T} = 0, \tag{2.36}$$

the (2,1) torsion $\mathbf{T}$ is holomorphic and therefore it determines a class in $H_{\bar{\partial}}^{2,1}(M)$. In local coordinates (2.36) has the form

$$\partial_{\bar{\mu}} T_{\alpha\bar{\lambda}\beta} = \partial_{\bar{\lambda}} T_{\alpha\bar{\mu}\beta}. \tag{2.37}$$

On the other hand

$$\begin{aligned}
D^\alpha \theta_\alpha = g^{\alpha\mu}\partial_\mu\theta_\alpha &= g^{\alpha\mu}\partial_{\bar{\mu}}(g^{\beta\bar{\lambda}} T_{\beta\bar{\lambda}\alpha}) \\
&= g^{\alpha\mu}\partial_{\bar{\mu}} g^{\beta\bar{\lambda}} . T_{\beta\bar{\lambda}\alpha} + g^{\alpha\bar{\mu}} g^{\beta\bar{\lambda}} \partial_{\bar{\mu}} T_{\beta\bar{\lambda}\alpha} \\
&= -g^{\alpha\mu} g^{\beta\bar{\sigma}} \Gamma_{\bar{\mu}}{}^{\bar{\lambda}}{}_{\bar{\sigma}} T_{\beta\bar{\lambda}\alpha} + g^{\alpha\bar{\mu}} g^{\beta\bar{\lambda}} \partial_{\bar{\lambda}} T_{\beta\bar{\mu}\alpha} \\
&= -g^{\alpha\mu} g^{\beta\lambda} \Gamma_\mu{}^{\bar{\sigma}}{}_\lambda T_{\beta\bar{\sigma}\alpha} + g^{\alpha\bar{\mu}} g^{\beta\lambda} D_{\bar{\lambda}} T_{\beta\bar{\mu}\alpha} + g^{\alpha\mu} g^{\beta\lambda} \Gamma_{\bar{\lambda}}{}^{\bar{\sigma}}{}_{\bar{\mu}} T_{\beta\bar{\sigma}\alpha} \\
&= -D^\beta\theta_\beta + g^{\alpha\mu} g^{\beta\bar{\lambda}} (\Gamma_{\bar{\lambda}}{}^{\bar{\sigma}}{}_\mu - \Gamma_{\bar{\mu}}{}^{\bar{\sigma}}{}_{\bar{\lambda}}) T_{\beta\bar{\sigma}\alpha} \\
&= -D^\alpha\theta_\alpha + T^{\beta\bar{\sigma}\alpha} T_{\beta\bar{\sigma}\alpha}.
\end{aligned}$$

In fourth equality we used (2.37). Hence

$$2D^\alpha\theta_\alpha = |T|^2. \tag{2.38}$$

(2.24) and (2.38) imply

$$2|\theta|^2 + 2\delta'\theta = |T|^2. \tag{2.39}$$

Integrate (2.39):

$$2\|\theta\|_{L_2} = \|T\|_{L_2}. \tag{2.40}$$

Therefore among all metrics which satisfy (2.35) or (2.36) none is balanced. Otherwise (2.40) will give $T = 0$,which is impossible on $\#_n S^3 \times S^3$ as we have seen several times. Also there are no metrics in C for which (2.35) holds. Indeed, if a metric is in C, it is easy to see that $\|\theta\|_{L_2} = \|T\|_{L_2}$, which together with (2.40) implies $T = \theta = 0$.

We note also that (2.35) means the invariant $K_1 = 0$([24]).

## 2.9 Deformations of the Hermitian-Einstein metric

Suppose that $g$ is an Hermitian-Einstein metric with zero Einstein factor on a compact Hermitian manifold with trivial canonical bundle. For instance, $g$ could be the Hermitian-Einstein metric on $M$, which we deduced in section 2.5. In this section we shall obtain the corresponding Lichnerowicz equation for the deformations of $g$ keeping the complex structure and the volume fixed. Further, following [36],we shall get that if $g$ is in the class C, then there are no non-zero essential Hermitian deformations of $g$.

Let $g_t$ be a 1-parameter family of Hermitian-Einstein metrics on $M$ and

$$h = \frac{d}{dt}|_{t=0} g_t, \qquad g_0 = g.$$

be the infinitesimal deformation induced by $g_t$. We normalize the volume :

$$Vol(M) = \int_M \det(g_t) dz^1 \wedge \ldots \wedge d\bar{z}^3 = 1. \tag{2.41}$$

Differentiate (2.41) :

$$\int_M \det(g) g^{\lambda\bar{\mu}} h_{\lambda\bar{\mu}} dz^1 \wedge \ldots \wedge d\bar{z}^3 = 0.$$

Therefore

$$\int_M tr_g h = 0, \tag{2.42}$$

Differentiating the Hermitian-Einstein condition (2.21), we obtain

$$\frac{d}{dt}|_{t=0}g^{\lambda\mu}.R_{\alpha}{}^{\beta}{}_{\lambda\mu} + g^{\lambda\mu}\frac{d}{dt}|_{t=0}R_{\alpha}{}^{\beta}{}_{\lambda\mu} = 0.$$

We have

$$\frac{d}{dt}|_{t=0}g^{\lambda\mu} = -h^{\lambda\mu}.$$

Since $\Gamma$'s are the components of the Chern connection

$$\frac{\partial}{\partial z^{\lambda}}g_{\alpha\gamma} = g_{\beta\gamma}\Gamma_{\lambda}{}^{\beta}{}_{\alpha}.$$

Differentiate the above equation:

$$\frac{\partial}{\partial z^{\lambda}}h_{\alpha\gamma} = h_{\beta\gamma}\Gamma_{\lambda}{}^{\beta}{}_{\alpha} + g_{\beta\gamma}\frac{d}{dt}|_{t=0}\Gamma_{\lambda}{}^{\beta}{}_{\alpha}$$

and therefore

$$D_{\lambda}h_{\alpha\gamma} = g_{\beta\gamma}\frac{d}{dt}|_{t=0}\Gamma_{\lambda}{}^{\beta}{}_{\alpha},$$

i.e.

$$\frac{d}{dt}|_{t=0}\Gamma_{\lambda}{}^{\beta}{}_{\alpha} = D_{\lambda}h_{\alpha}{}^{\beta} = g^{\beta\mu}D_{\lambda}h_{\alpha\mu}.$$

It follows that

$$\frac{d}{dt}|_{t=0}R_{\alpha}{}^{\beta}{}_{\lambda\mu} = g^{\beta\sigma}D_{\mu}D_{\lambda}h_{\alpha\sigma} = D_{\mu}D_{\lambda}h_{\alpha}{}^{\beta}.$$

Therefore the deformations satisfy the Lichnerowicz equation

$$D^{\lambda}D_{\lambda}h_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta}{}_{\lambda\mu}h^{\lambda\mu}. \qquad (2.43)$$

Take the trace in (2.43) :

$$D^{\lambda}D_{\lambda}(tr_g h) = 0$$

for (2.20). By the maximum principle of E.Hopf

$$tr_g h = c = const.$$

Inserting this in (2.42) gives

$$tr_g h = 0.$$

Therefore the infinitesimal deformations of the metric satisfy

$$\begin{cases} \triangle_L h = 0 \\ tr_g h = 0, \end{cases} \tag{2.44}$$

where

$$(\triangle_L h)_{\alpha\bar{\beta}} = -D^{\lambda} D_{\lambda} h_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}}{}^{\tau\sigma} h_{\sigma\tau}$$

is the corresponding Lichnerowicz laplacian.

The space $HETID$ of all trivial infinitesimal Hermitian-Einstein deformations of $g$ consists of deformations of the form $L_X g = \delta_g^* X$, where $L_X$ is the Lie derivative ([36]). The space $HEEID$ of the essential Hermitian-Einstein deformations of $g$ is the orthogonal space to $HETID$ with respect to the global product determined by $g$. We shall calculate $HEEID$.

By definition

$$(L_X g)(Y, Z) = Xg(Y, Z) - g(L_X Y, Z)$$

or in local coordinates

$$(L_X g)_{\lambda\mu} = X^{\sigma} \frac{\partial}{\partial z^{\sigma}} g_{\lambda\mu} + g_{\sigma\mu} \frac{\partial X^{\sigma}}{\partial z^{\lambda}}.$$

Thus

$$\begin{aligned}
0 = < h_{\alpha\bar{\beta}}, (L_X g)_{\alpha\bar{\beta}} > &= \int_M h_{\alpha\bar{\beta}} g^{\alpha\bar{\mu}} g^{\lambda\bar{\beta}} (L_X g)_{\lambda\bar{\mu}} \\
&= \int_M h_{\alpha\bar{\beta}} g^{\alpha\bar{\mu}} g^{\lambda\bar{\beta}} [X^{\sigma} \frac{\partial}{\partial z^{\sigma}} g_{\lambda\bar{\mu}} + g_{\sigma\bar{\mu}} \frac{\partial X^{\sigma}}{\partial z^{\lambda}}] \\
&= \int_M [-\partial_{\lambda} h_{\alpha\bar{\beta}} g^{\alpha\mu} g^{\lambda\beta} g_{\sigma\mu}. X^{\sigma} - h_{\alpha\bar{\beta}} \partial_{\lambda} g^{\alpha\mu}. g^{\lambda\beta} g_{\sigma\bar{\mu}}. X^{\sigma} \\
&\quad - h_{\alpha\beta} \partial_{\lambda} g^{\lambda\beta}. X^{\alpha} - h_{\alpha\bar{\beta}} g^{\alpha\mu} g^{\lambda\beta} \partial_{\lambda} g_{\sigma\mu}. X^{\sigma} + h_{\alpha\bar{\beta}} g^{\alpha\bar{\mu}} g^{\lambda\beta} \partial_{\sigma} g_{\lambda\bar{\mu}}. X^{\sigma}] \\
&= \int_M [(-\partial_{\lambda} h_{\alpha\bar{\beta}} + \Gamma_{\lambda}{}^{\sigma}{}_{\alpha} h_{\sigma\bar{\beta}}) g^{\lambda\beta} X^{\alpha} + \Theta^{\beta} h_{\alpha\bar{\beta}} X^{\alpha} + T_{\sigma}{}^{\alpha}{}_{\lambda} h_{\alpha\bar{\beta}} g^{\lambda\bar{\beta}} X^{\sigma}] \\
&= \int_M [-D^{\beta} h_{\alpha\bar{\beta}} + \Theta^{\beta} h_{\alpha\bar{\beta}} + T_{\alpha}{}^{\lambda\beta} h_{\lambda\bar{\beta}}] X^{\alpha}.
\end{aligned}$$

Therefore

$$D^{\alpha} h_{\alpha\bar{\beta}} - \Theta^{\alpha} h_{\alpha\bar{\beta}} - T_{\bar{\beta}}{}^{\tau\sigma} h_{\sigma\tau} = 0 \tag{2.45}$$

(2.44) and (2.45) determine $HEEID$.

In order to investigate the Hermitian deformations we need a commutation formula of Weitzenböck type. We have

$$D^{\alpha}D_{\beta}h_{\alpha\gamma} = g^{\alpha\bar{\mu}}D_{\bar{\mu}}[D_{\beta}h_{\alpha\bar{\gamma}}]$$

$$= g^{\alpha\bar{\mu}}[\partial_{\mu}(D_{\beta}h_{\alpha\bar{\gamma}}) - \Gamma_{\bar{\mu}}{}^{\bar{\sigma}}{}_{\bar{\gamma}}D_{\beta}h_{\alpha\bar{\sigma}}]$$

$$= g^{\alpha\bar{\mu}}[\partial_{\mu}\partial_{\beta}h_{\alpha\bar{\gamma}} - \partial_{\bar{\mu}}\Gamma_{\beta}{}^{\tau}{}_{\alpha}h_{\tau\bar{\gamma}} - \Gamma_{\beta}{}^{\tau}{}_{\alpha}\partial_{\mu}h_{\tau\gamma}$$

$$+\partial_{\beta}\Gamma_{\mu}{}^{\bar{\sigma}}{}_{\bar{\gamma}}h_{\alpha\bar{\sigma}} - \partial_{\beta}(\Gamma_{\mu}{}^{\bar{\sigma}}{}_{\bar{\gamma}}h_{\alpha\bar{\sigma}}) + \Gamma_{\bar{\mu}}{}^{\bar{\sigma}}{}_{\bar{\gamma}}\Gamma_{\beta}{}^{\tau}{}_{\alpha}h_{\tau\bar{\sigma}}]$$

$$= g^{\alpha\bar{\mu}}[\partial_{\beta}(\partial_{\mu}h_{\alpha\bar{\gamma}} - \Gamma_{\mu}{}^{\bar{\sigma}}{}_{\bar{\gamma}}h_{\alpha\bar{\sigma}}) - \Gamma_{\beta}{}^{\tau}{}_{\alpha}(\partial_{\mu}h_{\tau\bar{\gamma}} - \Gamma_{\mu}{}^{\bar{\sigma}}{}_{\bar{\gamma}}h_{\tau\bar{\sigma}})$$

$$-R_{\alpha}{}^{\tau}{}_{\beta\mu}h_{\tau\bar{\gamma}} + R_{\bar{\gamma}}{}^{\bar{\sigma}}{}_{\mu\beta}h_{\alpha\bar{\sigma}}]$$

$$= D_{\beta}D^{\alpha}h_{\alpha\bar{\gamma}} - R_{\alpha}{}^{\sigma}{}_{\beta\mu}g^{\alpha\bar{\mu}}h_{\sigma\bar{\gamma}} + R_{\bar{\gamma}}{}^{\bar{\sigma}}{}_{\bar{\mu}\beta}g^{\alpha\mu}h_{\alpha\bar{\sigma}}$$

$$= D_{\beta}D^{\alpha}h_{\alpha\bar{\gamma}} + (D^{\alpha}T_{\alpha}{}^{\sigma}{}_{\beta})h_{\sigma\bar{\gamma}} + (R_{\beta\bar{\gamma}}{}^{\bar{\sigma}\alpha} - D^{\alpha}T^{\bar{\sigma}}{}_{\bar{\gamma}\beta})h_{\alpha\bar{\sigma}}.$$

since by Bianchi's identity and (2.21)

$$g^{\alpha\bar{\mu}}R_{\beta}{}^{\sigma}{}_{\alpha\mu} - g^{\alpha\bar{\mu}}R_{\alpha}{}^{\sigma}{}_{\beta\mu} = -(D_{\mu}T_{\beta}{}^{\sigma}{}_{\alpha})g^{\alpha\bar{\mu}},$$

$$-g^{\alpha\bar{\mu}}R_{\alpha}{}^{\sigma}{}_{\beta\mu} = D^{\alpha}T_{\alpha}{}^{\sigma}{}_{\beta}$$

and also since

$$R_{\bar{\gamma}}{}^{\bar{\sigma}}{}_{\mu\beta} = g^{\nu\bar{\sigma}}R_{\bar{\gamma}\nu\mu\beta} = g^{\nu\bar{\sigma}}R_{\nu\bar{\gamma}\beta\mu}$$

$$= g^{\nu\bar{\sigma}}[R_{\beta\bar{\gamma}\nu\bar{\mu}} - D_{\mu}T_{\nu\bar{\gamma}\beta}],$$

that is,

$$R_{\bar{\gamma}}{}^{\bar{\sigma}}{}_{\mu\beta} = R_{\beta\bar{\gamma}}{}^{\bar{\sigma}}{}_{\bar{\mu}} - D^{\alpha}T^{\sigma}{}_{\bar{\gamma}\beta}.$$

Therefore

$$-D^{\alpha}(D_{\alpha}h_{\beta\bar{\gamma}} - D_{\beta}h_{\alpha\bar{\gamma}}) = -D^{\alpha}D_{\alpha}h_{\beta\bar{\gamma}} + R_{\beta\bar{\gamma}}{}^{\bar{\sigma}\alpha}h_{\alpha\bar{\sigma}} + D_{\beta}D^{\alpha}h_{\alpha\gamma}$$

$$+ (D^{\alpha}T_{\alpha}{}^{\sigma}{}_{\beta})h_{\sigma\bar{\gamma}} - (D^{\alpha}T^{\bar{\sigma}}{}_{\bar{\gamma}\beta})h_{\alpha\bar{\sigma}}. \tag{2.46}$$

Now suppose that the Hermitian-Einstein metric is in C. We are going to rephrase (2.46) for this particular case. Recall that by definition $g \in$C if and only if

$$T_{\alpha}{}^{\sigma}{}_{\beta} = \frac{1}{2}(\theta_{\beta}\delta_{\alpha}^{\sigma} - \theta_{\alpha}\delta_{\beta}^{\sigma}). \tag{2.47}$$

Then

$$D^\alpha T_\alpha{}^\sigma{}_\beta = \frac{1}{2}[\delta^\sigma_\alpha D^\alpha\theta_\beta - \delta^\sigma_\beta D^\alpha\theta_\alpha]$$
$$= \frac{1}{2}[D^\sigma\theta_\beta - \delta^\sigma_\beta v]$$

by (2.47) and therefore

$$(D^\alpha T_\alpha{}^\sigma{}_\beta)h_{\sigma\gamma} = \frac{1}{2}(D^\sigma\theta_\beta)h_{\sigma\gamma} - \frac{1}{2}v h_{\beta\gamma} \tag{2.48}$$

Moreover

$$T^\tau{}_{\gamma\beta} = g_{\mu\gamma}g^{\nu\tau}T_\nu{}^\mu{}_\beta = \frac{1}{2}(\theta_\beta\delta^\mu_\nu - \theta_\nu\delta^\mu_\beta)g_{\mu\gamma}g^{\nu\tau}$$
$$= \frac{1}{2}\theta_\beta\delta^\tau_\gamma - \frac{1}{2}\Theta^\tau g_{\beta\gamma}.$$

Hence

$$D^\sigma T^\tau{}_{\gamma\beta} = \frac{1}{2}\delta^\tau_\gamma D^\sigma\theta_\beta - \frac{1}{2}g_{\beta\gamma}(D^\sigma\Theta^\tau).$$

Thus

$$(D^\sigma T^\tau{}_{\gamma\beta})h_{\sigma\tau} = \frac{1}{2}(D^\sigma\theta_\beta)h_{\sigma\gamma} - \frac{1}{2}g_{\beta\gamma}(D^\sigma\Theta^\tau)h_{\sigma\tau}. \tag{2.49}$$

Subtract (2.49) from (2.48):

$$(D^\alpha T_\alpha{}^\sigma{}_\beta)h_{\sigma\gamma} - (D^\sigma T^\tau{}_{\gamma\beta})h_{\sigma\tau} = -\frac{1}{2}v h_{\beta\gamma} + \frac{1}{2}g_{\beta\gamma}(D^\sigma\Theta^\tau)h_{\sigma\tau}.$$

Hence and from (2.46) we see that if $g$ is in $\dot{\mathrm{C}}$, the following formula holds:

$$-D^\alpha(D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma}) = \triangle_L h_{\beta\gamma} + D_\beta D^\alpha h_{\alpha\gamma}$$
$$- \frac{1}{2}v h_{\beta\gamma} + \frac{1}{2}g_{\beta\gamma}(D^\sigma\Theta^\tau)h_{\sigma\tau}. \tag{2.50}$$

Differentiate (2.47) with respect to $t$ :

$$\frac{d}{dt}|_{t=0}T_\alpha{}^\sigma{}_\beta = g^{\sigma\gamma}(D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma})$$
$$= \frac{1}{2}(D^\mu h_{\beta\mu}\cdot\delta^\sigma_\alpha - D^\mu h_{\alpha\mu}\cdot\delta^\sigma_\beta)$$

and therefore

$$D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma} = \frac{1}{2}(D^\mu h_{\beta\mu})g_{\alpha\gamma} - \frac{1}{2}(D^\mu h_{\alpha\mu})g_{\beta\gamma}.$$

Denote

$$\varphi_\alpha = D^\mu h_{\alpha\mu}.$$

From above

$$D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma} = \frac{1}{2}\varphi_\beta g_{\alpha\gamma} - \frac{1}{2}\varphi_\alpha g_{\beta\gamma} \tag{2.51}$$

Then (2.51) and (2.52) give

$$\begin{aligned} -\frac{1}{2}D^\alpha(\varphi_\beta g_{\alpha\gamma} - \varphi_\alpha g_{\beta\gamma}) &= \triangle_L h_{\beta\gamma} + D_\beta\varphi_\gamma \\ &- \frac{1}{2}v h_{\beta\gamma} + \frac{1}{2}(D^\alpha\Theta^\sigma)g_{\beta\gamma}h_{\alpha\sigma}. \end{aligned} \tag{2.52}$$

Since $D$ is the Chern connection, $g$ is parallel with respect to $D$. Hence, (2.52) can be rewritten as

$$\begin{aligned} -\frac{1}{2}D_\gamma\varphi_\beta + \frac{1}{2}(D^\alpha\varphi_\alpha)g_{\beta\gamma} &= \triangle_L h_{\beta\gamma} + D_\beta\varphi_\gamma \\ &- \frac{1}{2}v h_{\beta\gamma} + \frac{1}{2}(D^\alpha\Theta^\sigma)g_{\beta\gamma}h_{\alpha\sigma}. \end{aligned} \tag{2.53}$$

Multiply (2.53) by $h^{\beta\gamma}$ and integrate

$$-\frac{1}{2} < D_\gamma\varphi_\beta, h_{\beta\gamma} > = < \triangle_L h_{\beta\gamma}, h_{\beta\gamma} > + < D_\beta\varphi_\gamma, h_{\beta\gamma} > -\frac{1}{2}\int_M |\theta|^2|h|^2, \tag{2.54}$$

since $tr_g h = g^{\beta\gamma}h_{\beta\gamma} = g_{\beta\gamma}h^{\beta\gamma} = 0$ and we have chosen $g$ to be a Gauduchon metric.(As we mentioned in section 2.6, the class $W_4$ is conformally invariant and therefore C is also a conformally invariant class and there is a unique Gauduchon metric in it.)

Integrate (2.54) by parts :

$$\begin{aligned} -\frac{1}{2} < \varphi_\beta, \Theta^\lambda h_{\lambda\beta} > +\frac{1}{2} < \varphi_\beta, D^\lambda h_{\lambda\beta} > &= < \triangle_L h_{\beta\gamma}, h_{\beta\gamma} > \\ + < \varphi_\gamma, \Theta^\mu h_{\gamma\mu} > - < \varphi_\gamma, D^\mu h_{\gamma\mu} > &-\frac{1}{2}\int_M |\theta|^2|h|^2, \end{aligned}$$

or

$$\frac{3}{2} < \varphi_\beta, \varphi_\beta > +\frac{1}{2}\int_M |\theta|^2|h|^2 = < \triangle_L h, h > + < \varphi_\gamma, \Theta^\mu h_{\gamma\mu} > +\frac{1}{2} < \varphi_\beta, \Theta^\lambda h_{\lambda\beta} > . \tag{2.55}$$

Now suppose that $h$ is an essential deformation, that is, $\delta h = 0$. The latter is equivalent to (2.45). Since $g \in$C (see (2.47)):

$$T_\beta{}^{\tau\sigma}h_{\sigma\tau} = \frac{1}{2}g^{\sigma\lambda}\,\theta_\lambda\,\delta^\tau_\beta h_{\sigma\tau} - \frac{1}{2}g^{\sigma\lambda}\,\theta_\beta\,\delta^\tau_\lambda h_{\sigma\tau} = \frac{1}{2}\Theta^\sigma h_{\sigma\tau}$$

because $tr_g h = 0$. Therefore $h$ is essential if and only if

$$\varphi_{\bar\beta} = D^\alpha h_{\alpha\bar\beta} = \frac{3}{2}\Theta^\alpha h_{\alpha\bar\beta}. \tag{2.56}$$

Take complex conjugate of (2.56):

$$\varphi_\beta = \frac{3}{2}\Theta^{\bar\alpha} h_{\beta\bar\alpha}. \tag{2.57}$$

Now by (2.55), (2.56) and (2.57) we obtain

$$\frac{3}{2}\|\varphi\|^2 + \frac{1}{2}\int_M |\theta|^2|h|^2 =< \Delta_L h, h > + \frac{2}{3}\|\varphi\|^2 + \frac{1}{2}\cdot\frac{2}{3}\|\varphi\|^2,$$

that is,

$$\frac{1}{2}\|\varphi\|^2 + \frac{1}{2}\int_M |\theta|^2|h|^2 =< \Delta_L h, h > . \tag{2.58}$$

But $h$ is a deformation of an Hermitian-Einstein metric , i.e.

$$\Delta_L h = 0.$$

(See (2.45)). Since $\theta \neq 0$, (2.45) and (2.59) imply that

$$h = 0.$$

In this way we have obtained the following

**Proposition 2.6.** *If $g$ is both an Hermitian-Einstein metric with Einstein factor 0 and in the class C, then there are no non-zero essential Hermitian deformations of $g$.*

## 2.10 Deformations of the complex structure

Let introduce some notations taken from [36]. As in section 2.9 we shall follow again the exposition of Koiso.

$$CID = \{ \text{ complex infinitesimal deformations } \}$$

$$CTID(J) = \{ \text{ trivial } CID \} = \{ L_X J \}$$

$$CEID(g, J) = \{ \text{ essential } CID \} \perp CTID(J).$$

Let $J_t$ be a family of complex structures and $N_t$ be the corresponding Nijenhuis tensor. Denote

$$I = \frac{d}{dt}|_{t=0} J_t, \qquad J_0 = J.$$

Differentiating

$$J_t^2 = -id, \quad \text{and} \quad N_t = 0$$

we obtain as in [36], lemma 6.4, that $JI + IJ = 0$ and

$$I^{\alpha}{}_{\beta} = 0 \tag{2.59}$$

$$\partial_{\alpha} I^{\gamma}{}_{\beta} - \partial_{\beta} I^{\gamma}{}_{\alpha} = 0. \tag{2.60}$$

(2.59) and (2.60) determine the space $CID$. In terms of the metric $g$, (2.61) can be rewritten as

$$D_{\alpha} I^{\gamma}{}_{\beta} - D_{\beta} I^{\gamma}{}_{\alpha} + T_{\alpha}{}^{\sigma}{}_{\beta} I^{\gamma}{}_{\sigma} = 0.$$

In fact, we are more interested in the space $CEID$. The obvious orthogonal decomposition holds:

$$CID(J) = CEID(J) \oplus CTID(J).$$

Now we calculate $CEID$.

$$0 = < I^{\gamma}{}_{\alpha}, (L_X J)^{\gamma}{}_{\alpha} > = \int_M I^{\gamma}{}_{\alpha} (L_X J)^{\mu}{}_{\lambda} g^{\alpha\lambda} g_{\mu\gamma}$$

$$= 2i \int_M I_{\mu\alpha} (D_{\lambda} X^{\mu}) g^{\alpha\lambda} = 2i \int_M I_{\mu\alpha} (\partial_{\lambda} X_{\sigma} - \Gamma_{\lambda}{}^{\tau}{}_{\sigma} X_{\tau}) g^{\mu\sigma} g^{\alpha\lambda}.$$

We integrate by parts :

$$0 = -\int_M \partial_{\lambda} I_{\mu\alpha} . g^{\alpha\lambda} . X^{\mu} - \int_M I_{\mu\alpha} \partial_{\lambda} g^{\mu\sigma} . g^{\alpha\lambda} . X_{\sigma}$$

$$- \int_M I_{\mu\alpha} . X^{\mu} \partial_{\lambda} g^{\alpha\lambda} - \int_M I_{\mu\alpha} \Gamma_{\lambda}{}^{\tau}{}_{\sigma} . X_{\tau} g^{\mu\sigma} g^{\alpha\lambda}$$

$$= -\int_M [D^{\alpha} I_{\mu\alpha} - T_{\lambda}{}^{\lambda\alpha} I_{\mu\alpha}] X^{\mu}$$

since

$$\partial_{\lambda} g^{\alpha\lambda} = -D_{\lambda} g^{\alpha\lambda} + \Gamma_{\lambda}{}^{\lambda}{}_{\tau} g^{\alpha\tau}$$

$$= T_{\lambda}{}^{\lambda}{}_{,\sigma} g^{\alpha\sigma} + \Gamma_{\sigma}{}^{\lambda}{}_{\lambda} g^{\alpha\sigma} = T_{\lambda}{}^{\lambda\alpha}.$$

Therefore

$$D^{\alpha} I_{\mu\alpha} = \Theta^{\alpha} I_{\mu\alpha}.$$

**Lemma 2.8.** $CEID \cong H^1(M, \Theta)$.

**Proof.** Let $I \in CID$,

$$I = I^{\gamma}{}_{\beta} dz^{\beta} \otimes \frac{\partial}{\partial \bar{z}^{\gamma}}.$$

Define

$$\iota(I) = I \rfloor d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3$$

$$= I^1{}_{\beta} dz^{\beta} \wedge d\bar{z}^2 \wedge d\bar{z}^3 - I^2{}_{\beta} dz^{\beta} \wedge d\bar{z}^1 \wedge d\bar{z}^3 + I^3{}_{\beta} dz^{\beta} \wedge d\bar{z}^1 \wedge d\bar{z}^2.$$

Therefore $\iota(I) \in \wedge^{1,2}$. Then by (2.61)

$$\partial \iota(I) = \partial_{\gamma} I^1{}_{\beta} dz^{\gamma} \wedge dz^{\beta} \wedge d\bar{z}^2 \wedge d\bar{z}^3 - \ldots$$

$$= \partial_{\beta} I^1{}_{\gamma} dz^{\gamma} \wedge dz^{\beta} \wedge d\bar{z}^2 \wedge d\bar{z}^3 - \ldots = -\partial \iota(I).$$

Thus $\partial \iota(I) = 0$ and

$$\iota(I) \in H^{1,2}_{\partial} \cong H^{2,1}_{\bar{\partial}} = H^1(M, \Omega^2) = H^1(M, \Theta).$$

Hence

$$I \longrightarrow \iota(I)$$

provides

$$CID \longrightarrow H^1(M, \Theta)$$

What is Ker $\iota$ ? $\iota(I) = [0]$ if and only if $\iota(I) = \partial\varphi$ for $\varphi \in \wedge^{0,2}$. Write

$$\varphi = \varphi_{12} d\bar{z}^1 \wedge d\bar{z}^2 + \varphi_{13} d\bar{z}^1 \wedge d\bar{z}^3 + \varphi_{23} d\bar{z}^2 \wedge d\bar{z}^3.$$

Define $\eta$ by

$$I^1{}_{\beta} = \partial_{\beta} \varphi_{23} = \partial_{\beta} \eta^1$$

$$I^2{}_{\beta} = \partial_{\beta} \varphi_{13} = \partial_{\beta} \eta^2$$

$$\Gamma^3{}_\beta = \partial_\beta \varphi_{12} = \partial_\beta \eta^3,$$

that is,

$$\Gamma^\gamma{}_\beta = \partial_\beta \eta^\gamma.$$

Since $\dim_C M = 3$, $\eta$ is uniquely determined. Set

$$\xi = i\eta^\gamma \frac{\partial}{\partial \bar{z}^\gamma}.$$

Then

$$\begin{aligned}(L_\xi J)\frac{\partial}{\partial \bar{z}^\alpha} &= -2i(D_\alpha i\eta^\gamma)/2\frac{\partial}{\partial \bar{z}^\gamma} = g^{\sigma\gamma}(D_\alpha \eta_\sigma)\frac{\partial}{\partial \bar{z}^\gamma}\\ &= [g^{\sigma\gamma}\partial_\alpha \eta_\sigma - g^{\sigma\gamma}\Gamma_\alpha{}^\tau{}_\sigma \eta_\tau]\frac{\partial}{\partial \bar{z}^\gamma}\\ &= [\partial_\alpha(g^{\sigma\gamma}\eta_\sigma) - \eta_\sigma \partial_\alpha g^{\sigma\gamma} + g^{\tau\gamma}\Gamma_\alpha{}^\sigma{}_\tau \eta_\sigma]\frac{\partial}{\partial \bar{z}^\gamma}\\ &= \partial_\alpha \eta^\gamma \frac{\partial}{\partial \bar{z}^\gamma} = \Gamma^\gamma{}_\alpha \frac{\partial}{\partial \bar{z}^\gamma}.\end{aligned}$$

Therfore

$$I = L_\xi J.$$

that is,

$$\mathrm{Ker}\, \iota \cong CTID.$$

So $\iota$ gives an injection

$$CEID \longrightarrow H^1(M, \dot{\Theta}).$$

Let $\psi \in H^1(M, \Theta) \cong H_{\bar\partial}^{1,2}(M)$. Therefore $\bar\partial\psi = 0$ and

$$\partial_\sigma \psi_{\beta\alpha\gamma} = \partial_\beta \psi_{\sigma\alpha\gamma}.$$

Let $\bar{\mu}$ be such that $\bar{\mu} \neq \alpha \neq \gamma \neq \bar{\mu}$ and $\{\mu, \alpha, \gamma\} = \{1, 2, 3\}$. Then define

$$I^\mu{}_\beta = \psi_{\beta\alpha\gamma}.$$

But

$$\partial_\lambda I^\mu{}_\beta - \partial_\beta I^\mu{}_\lambda = \partial_\lambda \psi_{\beta\alpha\gamma} - \partial_\beta \psi_{\lambda\alpha\gamma} = 0.$$

Therefore $I \in CID$ and we have surjection.

## 2.11 On the deformations of both the Hermitian metric and the complex structure

Let

$$CEID = CEID_S \oplus CEID_A$$

be the decomposition of $CEID$ into symmetric and anti-symmetric parts. We are going to prove that if we vary both the Hermitian metric and the complex structure, then any compact Hermitian non-Kählerian manifold with $h^{2,0} = 0$ admits no non-zero anti-symmetric deformations of the complex structure.

**Proposition 2.7.** $CEID_A = 0$.

**Proof.** Let

$$a = g_t(T_t(X,Y), J_t Z).$$

Differentiate $a$ :

$$\dot{a} = h(T(X,Y), JZ) + g(T^0(X,Y), JZ) + g(T(X,Y), IZ).$$

On the other hand $a = b$, where

$$b = g_t(D^t_X Y - D^t_Y X - [X,Y], J_t Z)$$

$$= g_t(D^t_X Y, J_t Z) + g_t(J_t D^t_Y X, Z) - g_t([X,Y], J_t Z)$$

$$= Xg_t(Y, J_t Z) - g_t(Y, J_t D^t_X Z) + Yg_t(Z, J_t X)$$

$$-g_t(D^t_Y Z, J_t X) - g_t([X,Y], J_t Z).$$

Now we differentiate $b$ :

$$\dot{b} = Xh(Y, JZ) + Xg(Y, IZ) - h(Y, JD^t_X Z) - g(Y, JD^0_X Z)$$

$$-g(Y, ID_X Z) + Yh(Z, JX) + Yg(Z, IX) - h(D_Y Z, JX)$$

$$-g(D^0_Y Z, JX) - g(D_Y Z, IX) - h([X,Y], JZ) - g([X,Y], IZ).$$

But $\dot{a} = \dot{b}$ since $a = b$. Therefore in a local complex frame adapted to the complex structure we have

$$\dot{a}_{\alpha\beta\bar{\gamma}} = -ih_{\sigma\bar{\gamma}} T_\alpha{}^\sigma{}_\beta - ig_{\sigma\bar{\gamma}} \dot{T}_\alpha{}^\sigma{}_\beta$$

and

$$\dot{b}_{\alpha\beta\gamma} = -i\partial_\alpha h_{\beta\gamma} + i\partial_\beta h_{\alpha\gamma}.$$

In this way from $\dot{a}_{\alpha\beta\gamma} = \dot{b}_{\alpha\beta\gamma}$ we get that

$$\partial_\alpha h_{\beta\gamma} - \partial_\beta h_{\alpha\gamma} = h_{\sigma\gamma} T_\alpha{}^\sigma{}_\beta + g_{\sigma\gamma} \dot{T}_\alpha{}^\sigma{}_\beta$$

or

$$g_{\sigma\gamma} \dot{T}_\alpha{}^\sigma{}_\beta = D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma}. \tag{2.61}$$

Further

$$\begin{aligned} dF(X,Y,Z) = \frac{1}{2}[ & XF(Y,Z) + YF(Z,X) + ZF(X,Y) \\ & - F([X,Y],Z) - F([Y,Z],X) - F([Z,X],Y)]. \end{aligned} \tag{2.62}$$

By definition

$$T(X,Y) = D_X Y - D_Y X - [X,Y].$$

Since the Chern connection $D$ is metric (see (2.3))

$$Dg = 0$$

or equivalently,

$$Xg(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z).$$

Substitute $Z \to JZ$:

$$XF(Y,Z) = F(D_X Y, Z) + F(Y, D_X Z),$$

since $D$ preserves the complex structure (see (2.3)) and

$$F(X,Y) = -g(X, JY).$$

Therefore

$$\begin{aligned} 2dF(X,Y,Z) = & F(D_X Y, Z) + F(Y, D_X Z) + F(D_Y Z, X) \\ & + F(Z, D_Y X) + F(D_Z X, Y) + F(X, D_Z Y) \\ & - F([X,Y],Z) - F([Y,Z],X) - F([Z,X],Y) \end{aligned}$$

$$= F(D_XY - D_YX - [X,Y], Z) + F(D_ZX - D_XZ - [Z,X], Y) + F(D_YZ - D_ZY - [Y,Z], X)$$

$$= F(T(X,Y),Z) + F(T(Z,X),Y) + F(T(Y,Z),X)$$

because

$$F(U,V) = -F(V,U).$$

Therefore

$$dF(X,Y,Z) = \frac{1}{2}[F(T(X,Y),Z) + F(T(Z,X),Y) + F(T(Y,Z),X)]. \qquad (2.63)$$

Denote

$$\phi = \frac{d}{dt}|_{t=0}F_t$$

Then differentiate (2.63):

$$2d\phi(X,Y,Z) = \phi(T(X,Y),Z) + F(\dot{T}(X,Y),Z) + \phi(T(Y,Z),X)$$

$$+F(\dot{T}(Y,Z),X) + \phi(T(Z,X),Y) + F(\dot{T}(Z,X),Y).$$

Let

$$X = \partial_\alpha, \quad Y = \partial_\beta, \quad Z = \partial_\gamma.$$

Then

$$d\phi_{\alpha\beta\gamma} = [\phi_{\sigma\gamma}T_\alpha{}^\sigma{}_\beta + ig_{\sigma\gamma}\dot{T}_\alpha{}^\sigma{}_\beta]/2$$

$$= [ih_{\sigma\gamma}T_\alpha{}^\sigma{}_\beta + i(D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma})]/2$$

$$= i(D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma} + T_\alpha{}^\sigma{}_\beta h_{\sigma\gamma})/2. \qquad (2.64)$$

In (2.64) we used (2.61) and

$$\phi_{\sigma\gamma} = ih_{\sigma\gamma},$$

which can be obtained in the following way. First we differentiate $F_t(X,Y) = -g_t(X, J_tY)$. Then, taking $X = \partial_\alpha$ and $Y = \partial_\beta$ we get

$$\phi_{\sigma\gamma} = ih_{\sigma\gamma} - I_{\sigma\gamma}$$

Hence, the desired relation is a consequence from the fact that $I_{\sigma\bar\gamma} = 0$, or equivalently $I^{\alpha}{}_{\bar\beta} = 0$, see (2.59). If we differentiate (2.62), we shall obtain

$$2d\phi(X,Y,Z) = X\phi(Y,Z) + Y\phi(Z,X) + Z\phi(X,Y)$$

$$-\phi([X,Y],Z) - \phi([Y,Z],X) - \phi([Z,X],Y).$$

Put again

$$X = \partial_\alpha, \quad Y = \partial_\beta, \quad Z = \partial_{\bar\gamma}.$$

Then

$$d\phi_{\alpha\beta\bar\gamma} = [\partial_\alpha\phi_{\beta\bar\gamma} + \partial_\beta\phi_{\bar\gamma\alpha} + \partial_{\bar\gamma}\phi_{\alpha\beta}]/2$$

$$= i(D_\alpha h_{\beta\bar\gamma} - D_\beta h_{\alpha\bar\gamma} + T_\alpha{}^{\sigma}{}_\beta h_{\sigma\bar\gamma})/2 - \frac{1}{4}\partial_{\bar\gamma}(I_{\alpha\beta} - I_{\beta\alpha}). \tag{2.65}$$

Above we used

$$\phi_{\alpha\beta} = -ih_{\alpha\beta} - I_{\alpha\beta},$$

which can be obtained by differentiating $F_t(X,Y) = -g_t(X, J_tY)$. Then we differentiate

$$g_t(X,Y) = g_t(J_tX, J_tY).$$

The result gives

$$2ih_{\alpha\beta} + I_{\alpha\beta} + I_{\beta\alpha} = 0.$$

(See (2.68) below). Thus

$$\phi_{\alpha\beta} = -\frac{1}{2}(I_{\alpha\beta} - I_{\beta\alpha}).$$

Comparing (2.65) and (2.66) we obtain

$$\partial_{\bar\gamma}(I_{\alpha\beta} - I_{\beta\alpha}) = 0. \tag{2.66}$$

This is not interesting if $I \in CEID_S$. But if $I \in CEID_A$, then from (2.67) it follows that

$$\partial_{\bar\gamma} I_{\alpha\beta} = 0,$$

that is $\bar\partial I = 0$. Therefore $I$ is a holomorphic 2-form and hence it must be zero since $H^0(M, \Omega^2) = H_{\bar\partial}^{2,0}(M) = 0$. This comletes the proof.

From this proposition we see that

$$H^1(M,\Theta) = CEID = CEID_S,$$

i.e.

**Corollary 2.2.** *Infinitesimally the space of essential symmetric deformations of the complex structure coincides with* $H^1(M,\Theta)$.

We would like also to mention the following.

The metric $g_t$ is a Hermitian metric with respect to the complex structure $J_t$, i.e.

$$g_t(X,Y) = g_t(J_tX, J_tY).$$

Differentiate :

$$h(X,Y) = h(JX,JY) + g(IX,JY) + g(JX,IY). \tag{2.67}$$

From the obvious identity

$$h(X,Y) = \frac{1}{2}(h(X,Y) + h(JX,JY)) + \frac{1}{2}(h(X,Y) - h(JX,JY)),$$

we get that

$$h^H - h^A = h(JX,JY),$$

$$h^H + h^A = h(JX,JY) + K(X,Y) = h(X,Y),$$

where

$$h^H(X,Y) = \frac{1}{2}(h(X,Y) + h(JX,JY)),$$

$$h^A(X,Y) = \frac{1}{2}(h(X,Y) - h(JX,JY)),$$

$$K(X,Y) = g(IX,JY) + g(JX,IY).$$

Therefore

$$h^A = K/2$$

and

$$h^H = h(JX,JY) + K(X,Y)/2.$$

$h^A$ is the anti-Hermitian part of $h$ since

$$K(JX,JY) = g(IJX,J^2Y) + g(J^2X,IJY)$$

$$= -g(JIX,J^2Y) - g(J^2X,JIY)$$

$$= -g(IX,JY) - g(JX,IY) = -K(X,Y),$$

since $g$ is a Hermitian metric and

$$JI + IJ = 0,$$

which is obtained by differentiating $J_t^2 = -id$. $h^H$ is the Hermitian part of $h$ since

$$h^H(JX,JY) = h(J^2X,J^2Y) + \frac{1}{2}K(JX,JY)$$

$$= h(X,Y) - \frac{1}{2}K(X,Y)$$

$$= h(JX,JY) + K(X,Y) - \frac{1}{2}K(X,Y) = h^H(X,Y).$$

From (2.67) we obtain

$$2h_{\alpha\beta} = i(I_{\alpha\beta} + I_{\beta\alpha}).$$

Hence and by **Proposition 2.7** we get

$$h_{\alpha\beta} = iI_{\alpha\beta}$$

Therefore

$$h^A_{\alpha\beta} = iI_{\alpha\beta},$$

$$h^A_{\alpha\bar{\beta}} = 0$$

and

$$h^H_{\alpha\beta} = -h_{\alpha\beta} + iI_{\alpha\beta} = 0,$$

$$h^H_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}}.$$

**Proposition 2.8** *If $g$ has vanishing conformal torsion and if it is an Hermitian-Einstein metric on $M = \#_n S^3 \times S^3$, then the space $\varepsilon(M)$ of all such Hermitian-Einstein metrics on $M$ should have (real) dimension*

$$\dim \varepsilon(M) \leq 2(n-1).$$

**Proof.** We have the decomposition

$$\varepsilon(M) = HEEID_H \oplus HEEID_A$$

into Hermitian and anti-Hermitian parts. **Proposition 2.6** states that $HEEID_H = 0$. The relation $I_{\alpha\beta} = -ih_{\alpha\beta}$ implies that $HEEID_A$ is included in $CEID_S$. Hence and from **Corollary 2.2** it follows that

$$\dim \varepsilon(M) = \dim HEEID_A \leq \dim CEID_S = 2h^{2,1}$$

But $h^{2,1} = n - 1$ as we saw in the Introduction to this chapter. Therefore we obtain the desired estimate.

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